February 9 Math 2306 sec 59 Spring 2016

Section 6: Linear Equations Theory and Terminology

Recall that an n^{th} order linear IVP consists of an equation

$$a_n(x)\frac{d^ny}{dx^n} + a_{n-1}(x)\frac{d^{n-1}y}{dx^{n-1}} + \cdots + a_1(x)\frac{dy}{dx} + a_0(x)y = g(x)$$

to solve subject to conditions

$$y(x_0) = y_0, \quad y'(x_0) = y_1, \quad \ldots, \quad y^{(n-1)}(x_0) = y_{n-1}.$$

The problem is called **homogeneous** if $g(x) \equiv 0$. Otherwise it is called **nonhomogeneous.**

Theorem: Existence & Uniqueness

Theorem: If $a_0, \dots, a_n$ and g are continuous on an interval I, $a_n(x) \neq 0$ for each x in I, and x_0 is any point in I, then for any choice of constants $y_0, \dots, y_{n-1}$, the IVP has a unique solution $y(x)$ on I.

Put differently, we're guaranteed to have a solution exist, and it is the only one there is!

A Second Order Linear Boundary Value Problem

consists of a problem

$$a_2(x)\frac{d^2y}{dx^2} + a_1(x)\frac{dy}{dx} + a_0(x)y = g(x), \quad a < x < b$$

to solve subject to a pair of conditions[1]

$$y(a) = y_0, \quad y(b) = y_1.$$

However similar this is in appearance, the existence and uniqueness result **does not hold** for this BVP!

[1] Other conditions on y and/or y' can be imposed. The key characteristic is that conditions are imposed at both end points $x = a$ and $x = b$.

BVP Examples

All solutions of the ODE $y'' + 4y = 0$ are of the form

$$y = c_1 \cos(2x) + c_2 \sin(2x).$$

Solve the BVP

$$y'' + 4y = 0, \quad 0 < x < \frac{\pi}{4} \quad y(0) = 0, \quad y\left(\frac{\pi}{4}\right) = 0.$$

Apply $y(0) = 0$ $\quad y(0) = c_1 \cos 0 + c_2 \sin 0 = 0 \Rightarrow c_1 = 0$

Apply $y\left(\frac{\pi}{4}\right) = 0$ $\quad y\left(\frac{\pi}{4}\right) = 0 \cdot \cos\left(2 \cdot \frac{\pi}{4}\right) + c_2 \sin\left(2 \cdot \frac{\pi}{4}\right) = 0$

$c_2 \cdot 1 = 0 \Rightarrow c_2 = 0$

The problem has one solution $y = 0$.

BVP Examples

Solve the BVP

$$y'' + 4y = 0, \quad 0 < x < \pi \quad y(0) = 0, \quad y(\pi) = 0.$$

From $y(0)=0$, $c_1=0$ as before.

Apply $y(\pi)=0$: $\quad y(\pi) = c_2 \sin(2\cdot\pi) = 0$

$$c_2 \cdot 0 = 0$$ true for all real c_2

This problem is solvable.

It has infinitely many solutions

$y = c_2 \sin(2x)$ for any real c_2.

BVP Examples

Solve the BVP

$$y'' + 4y = 0, \quad 0 < x < \pi \quad y(0) = 0, \quad y(\pi) = 1.$$

Again $y(0)=0$ gives $c_1 = 0$.

Apply $y(\pi)=1$: $y(\pi) = c_2 \sin(2\cdot\pi) = 1$

$$c_2 \cdot 0 = 1$$

This is false for all c_2.

This problem has no solution.

Homogeneous Equations

We'll consider the equation

$$a_n(x)\frac{d^n y}{dx^n} + a_{n-1}(x)\frac{d^{n-1}y}{dx^{n-1}} + \cdots + a_1(x)\frac{dy}{dx} + a_0(x)y = 0$$

and assume that each a_i is continuous and a_n is never zero on the interval of interest.

Theorem: If $y_1, y_2, \ldots, y_k$ are all solutions of this homogeneous equation on an interval I, then the *linear combination*

$$y(x) = c_1 y_1(x) + c_2 y_2(x) + \cdots + c_k y_k(x)$$

is also a solution on I for any choice of constants $c_1, \ldots, c_k$.

This is called the **principle of superposition.**

Corollaries

(i) If y_1 solves the homogeneous equation, the any constant multiple $y = cy_1$ is also a solution.

(ii) The solution $y = 0$ (called the trivial solution) is always a solution to a homogeneous equation.

Big Questions:

- Does an equation have any **nontrivial** solution(s), and
- since y_1 and cy_1 aren't truly *different* solutions, what criteria will be used to call solutions distinct?

Linear Dependence

Definition: A set of functions $f_1(x), f_2(x), \ldots, f_n(x)$ are said to be **linearly dependent** on an interval I if there exists a set of constants $c_1, c_2, \ldots, c_n$ with at least one of them being nonzero such that

$$* \; c_1 f_1(x) + c_2 f_2(x) + \cdots + c_n f_n(x) = 0 \quad \text{for all} \quad x \text{ in } I.$$

A set of functions that is not linearly dependent on I is said to be **linearly independent** on I.

* This is called a linear dependence relation.

Example: A linearly Dependent Set

The functions $f_1(x) = \sin^2 x$, $f_2(x) = \cos^2 x$, and $f_3(x) = 1$ are linearly dependent on $I = (-\infty, \infty)$.

Recall $\quad \sin^2 x + \cos^2 x = 1 \quad$ for all real x

Note $\quad \sin^2 x + \cos^2 x - 1 = 0$

This is $\quad c_1 f_1(x) + c_2 f_2(x) + c_3 f_3(x) = 0$

where $\quad c_1 = 1, \; c_2 = 1 \;$ and $\; c_3 = -1$.

We have a set of c's where at least one is not zero.

So our functions are linearly dependent on $(-\infty, \infty)$.

Example: A linearly Independent Set

The functions $f_1(x) = \sin x$ and $f_2(x) = \cos x$ are linearly independent on $I = (-\infty, \infty)$.

We'll show that $c_1 f_1(x) + c_2 f_2(x) = 0$ for all x <u>only</u> works if $c_1 = c_2 = 0$.

Suppose $c_1 \sin x + c_2 \cos x = 0$ for all real x.

This has to hold when $x = 0$.

$$c_1 \sin 0 + c_2 \cos 0 = 0 \Rightarrow c_1 \cdot 0 + c_2 \cdot 1 = 0$$

i.e. $c_2 = 0$

The relation must hold @ $x = \pi/2$

$$c_1 \sin \pi/2 + 0 \cdot \cos \pi/2 = 0 \implies c_1 \cdot 1 = 0$$

i.e. $c_1 = 0$

$c_1 \sin x + c_2 \cos x = 0$ for all real x <u>only</u> if

$c_1 = 0$ and $c_2 = 0$.

Hence $f_1(x), f_2(x)$ are <u>linearly</u> <u>independent</u>.

Determine if the set is Linearly Dependent or Independent

$$f_1(x) = x^2, \quad f_2(x) = 4x, \quad f_3(x) = x - x^2 \qquad I = (-\infty, \infty)$$

A linear dependence relation would look like

$$c_1 f_1(x) + c_2 f_2(x) + c_3 f_3(x) = 0 \quad \text{for all } x$$

$$c_1 x^2 + c_2(4x) + c_3(x - x^2) = 0$$

$$c_1 x^2 + 4c_2 x + c_3 x - c_3 x^2 = 0$$

Try $c_1 = 3$ set $c_3 = 3$ and $c_2 = \frac{-3}{4}$

Then note that at least one is nonzero.

$$c_1 f_1(x) + c_2 f_2(x) + c_3 f_3(x) =$$

$$3x^2 + 4\left(\frac{-3}{4}\right)x + 3(x - x^2) =$$

$$3x^2 - 3x + 3x - 3x^2 = 0$$

We have a linear dependence relation.

Hence f_1, f_2, f_3 are <u>linearly</u> <u>dependent</u>.

Definition of Wronskian

Let f_1, $f_2, \ldots, f_n$ posses at least $n-1$ continuous derivatives on an interval I. The **Wronskian** of this set of functions is the determinant

$$W(f_1, f_2, \ldots, f_n)(x) = \begin{vmatrix} f_1 & f_2 & \cdots & f_n \\ f_1' & f_2' & \cdots & f_n' \\ \vdots & \vdots & \vdots & \vdots \\ f_1^{(n-1)} & f_2^{(n-1)} & \cdots & f_n^{(n-1)} \end{vmatrix}.$$

(Note that, in general, this Wronskian is a function of the independent variable x.)

Determine the Wronskian of the Functions

$$f_1(x) = \sin x, \quad f_2(x) = \cos x$$

$$f_1'(x) = \cos x \quad , \quad f_2'(x) = -\sin x$$

$$W(f_1, f_2)(x) = \begin{vmatrix} f_1(x) & f_2(x) \\ f_1'(x) & f_2'(x) \end{vmatrix} = \begin{vmatrix} \sin x & \cos x \\ \cos x & -\sin x \end{vmatrix}$$

$$= \sin x(-\sin x) - \cos x(\cos x)$$

$$= -\sin^2 x - \cos^2 x = -(\sin^2 x + \cos^2 x) = -1$$

Determinant of 2x2 : $\begin{vmatrix} a & b \\ c & d \end{vmatrix} = ad - cb$

3x3 $\begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix}$

$$= a_{11} \begin{vmatrix} a_{22} & a_{23} \\ a_{32} & a_{33} \end{vmatrix} - a_{12} \begin{vmatrix} a_{21} & a_{23} \\ a_{31} & a_{33} \end{vmatrix} + a_{13} \begin{vmatrix} a_{21} & a_{22} \\ a_{31} & a_{32} \end{vmatrix}$$

Determine the Wronskian of the Functions

$$f_1(x) = x^2, \quad f_2(x) = 4x, \quad f_3(x) = x - x^2$$

$f_1'(x) = 2x \qquad f_2'(x) = 4 \qquad f_3'(x) = 1 - 2x$

$f_1''(x) = 2 \qquad f_2''(x) = 0 \qquad f_3''(x) = -2$

$$W(f_1, f_2, f_3)(x) = \begin{vmatrix} f_1 & f_2 & f_3 \\ f_1' & f_2' & f_3' \\ f_1'' & f_2'' & f_3'' \end{vmatrix}$$

$$= \begin{vmatrix} x^2 & 4x & x-x^2 \\ 2x & 4 & 1-2x \\ 2 & 0 & -2 \end{vmatrix}$$

$$= x^2 \begin{vmatrix} 4 & 1-2x \\ 0 & -2 \end{vmatrix} - 4x \begin{vmatrix} 2x & 1-2x \\ 2 & -2 \end{vmatrix} + (x-x^2) \begin{vmatrix} 2x & 4 \\ 2 & 0 \end{vmatrix}$$

$$= x^2(-8-0) - 4x\left(-4x - 2(1-2x)\right) + (x-x^2)(-8)$$

$$= -8x^2 - 4x\left(-\cancel{4x} - 2 + \cancel{4x}\right) - 8x + 8x^2$$

$$= -8x^2 + 8x - 8x + 8x^2 = 0$$

$$W(f_1, f_2, f_3)(x) = 0$$

Theorem (a test for linear independence)

Let $f_1, f_2, \ldots, f_n$ be $n-1$ times continuously differentiable on an interval I. If there exists x_0 in I such that $W(f_1, f_2, \ldots, f_n)(x_0) \neq 0$, then the functions are **linearly independent** on I.

If $y_1, y_2, \ldots, y_n$ are n solutions of the linear homogeneous n^{th} order equation on an interval I, then the solutions are **linearly independent** on I if and only if $W(y_1, y_2, \ldots, y_n)(x) \neq 0$ for[2] each x in I.

[2]For solutions of one linear homogeneous ODE, the Wronskian is either always zero or is never zero.

Determine if the functions are linearly dependent or independent:

$y_1 = e^x, \quad y_2 = e^{-2x} \quad I = (-\infty, \infty)$

We'll use the Wronskian

$$W(y_1, y_2)(x) = \begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix} = \begin{vmatrix} e^x & e^{-2x} \\ e^x & -2e^{-2x} \end{vmatrix}$$

$$= e^x(-2e^{-2x}) - e^x(e^{-2x})$$

$$= -2e^{-x} - e^{-x} = -3e^{-x}$$

$$W(e^{x}, e^{-2x})(x) = -3e^{-x}$$

This is never zero.* Hence y_1, y_2 are linearly <u>independent</u>.

* It's sufficient that $W(x) \neq 0$ at at least one x-value.