Fourier Series
Calculus II Project

The purpose of this project is to explore a means of expressing a periodic function in the form of a series of *basic* periodic functions. We know that the functions $\cos x$ and $\sin x$ are periodic with period 2π. And, although 2π is not the *fundamental* period of functions of the form $\cos nx$ or $\sin nx$ (for integer n), these are also 2π-periodic as is the sum of any such functions.

Suppose we have a function f defined on the interval $[-\pi, \pi]$ and either not defined outside of this interval or that is 2π-periodic. We may pose the question: *Can we write f as a sum of functions of the form $\cos nx$ and $\sin nx$?* If the answer is "yes", then it should be that

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} (a_n \cos nx + b_n \sin nx) \tag{1}$$

for some collection of constants a_0, a_n, and b_n (taking $a_0/2$ is a convention that will make sense with a little bit of context). When such an expression can be found, the right hand side of (1) is called the *Fourier series* of f. If we have reason to believe that such a series exists, there are several important questions to ask. First among them would be "what are the coefficients?"

Carry out the following activities.

A. Let f and g be a pair of functions that are integrable on an interval $[a, b]$. We define the *inner product* of f and g, denoted $< f, g >$, by

$$< f, g >= \int_a^b f(x)g(x)\, dx.$$

(This is analogous to the dot product of vectors in $\mathbb{R}^2$.) Show that this inner product has the following properties:

i $< f, g >=< g, f >$.

ii For constant k, $< kf, g >= k < f, g >$.

iii For functions f, g, and h, $< f, g+h >=< f, g > + < f, h >$. (Here, $(g+h)(x) = g(x)+h(x)$ is standard addition of functions.)

iv $< f, f > \geq 0$ and $< f, f >= 0$ if and only if $f(x) = 0$ for every x in $[a, b]$.

B. A pair of functions f and g are called *orthogonal* if $< f, g >= 0$. A family of functions $\{\phi_n \,|\, n = 0, 1, 2, \ldots\}$ is called an orthogonal family if

$$< \phi_n, \phi_m >= 0 \quad \text{whenever} \quad n \neq m.$$

It is worth noting that orthogonality depends on the functions as well as the interval under consideration.

Suppose we take the interval $[-\pi, \pi]$ and consider the family of functions

$$\{\cos(nx), \sin(mx) \,|\, n = 0, 1, 2, \ldots, \; m = 1, 2, 3, \ldots\}.$$

This family includes the constant function $\cos(0x) = 1$.

Show that this is an orthogonal family of functions on $[-\pi, \pi]$. Note that you will have to consider several cases including a constant times a sine, a constant times a cosine, products of two sines, products of two cosines, and a product of a sine and a cosine. There are also infinitely many functions to consider, so specifying values of n and m (other than the $n = 0$ case) is not a reasonable strategy. Find or derive appropriate trigonometric identities to carry out the integration.

C. Now, suppose that f is defined on $[-\pi, \pi]$ and is either not defined outside of this interval or is 2π-periodic. We wish to write f in the form of an infinite sum

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} \left(a_n \cos(nx) + b_n \sin(nx)\right).$$

Let's suppose for now that this is truly an equality for every x in $[-\pi, \pi]$. The task is to determine what the (infinite number of) coefficients should be.

As a single example, let's find b_5 the coefficient of $\sin(5x)$ in this sum. Multiply both sides of this sum by $\sin(5x)$ to get

$$f(x)\sin(5x) = \frac{a_0}{2}\sin(5x) + \sum_{n=1}^{\infty} \left(a_n \cos(nx)\sin(5x) + b_n \sin(nx)\sin(5x)\right).$$

Next, let us integrate both sides with respect to x on the interval $[-\pi, \pi]$ under the assumption that we may interchange the operations of summation and integration.

$$\int_{-\pi}^{\pi} f(x)\sin(5x)\,dx = \int_{-\pi}^{\pi} \frac{a_0}{2}\sin(5x)\,dx + \sum_{n=1}^{\infty}\left(a_n \int_{-\pi}^{\pi} \cos(nx)\sin(5x)\,dx + b_n \int_{-\pi}^{\pi} \sin(nx)\sin(5x)\,dx\right).$$

In light of the orthogonality property established in part **B.**, the above reduces to

$$\int_{-\pi}^{\pi} f(x)\sin(5x)\,dx = b_5 \int_{-\pi}^{\pi} \sin^2(5x)\,dx = \pi b_5 \quad \Longrightarrow \quad b_5 = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\sin(5x)\,dx.$$

Generalize this result to find formulas for each of a_0, a_n, and b_n for every $n \geq 1$. Based on your findings, explain why many authors choose the constant term to be expressed as $\frac{a_0}{2}$ as opposed to just

a_0.

D. Consider the triangle wave function

$$f(x) = \begin{cases} \pi + x, & -\pi \le x \le 0 \\ \pi - x, & 0 < x \le \pi \end{cases}, \qquad f(x + 2\pi) = f(x).$$

Find the Fourier series for f. Using appropriate technology, plot f along with some partial sums of its Fourier series (e.g. taking the first 2, 5, 10 terms) together.

Find the Fourier series of the function defined on $[-\pi, \pi]$

$$f(x) = \begin{cases} \pi, & -\pi \le x \le 0 \\ x, & 0 < x \le \pi \end{cases}.$$

Plot this function. Also plot several partial sums of its Fourier series. Even though f is not defined outside of the interval $[-\pi, \pi]$, plot at least one partial sum (at least 10 nonzero terms) on the interval $[-5\pi, 5\pi]$. What do you notice?

E. A notable application of Fourier series is in solving differential equations for systems subject to periodic driving forces. Examples would include a spring mass system subject to a driving piston, or a biological network system subjected to periodic neural impulses.

Consider the following differential equation

$$\frac{d^2y}{dx^2} + 2y = \sum_{n=1}^{\infty} \frac{\sin(nx)}{n^2}, \qquad -\pi < x < \pi. \tag{2}$$

Assume that a solution $y = f(x)$ has a series representation of the form

$$f(x) = \sum_{n=1}^{\infty} B_n \sin(nx).$$

Solve the differential equation to find the coefficients B_n.

For the function $f(x)$ you found above, show that

$$y = f(x) + c_1 \cos(\sqrt{2}x) + c_2 \sin(\sqrt{2}x) \tag{3}$$

also solves (2). The function y in (3) is called the *general solution* of equation (2).

F. Find some reference material on the history and applications of Fourier series and extensions to the cases considered in the previous steps. Discuss what you find. Your discussion should address:

- If f is symmetric (even or odd) what immediate conclusions can be drawn about its Fourier series?
- How are the coefficient formulas generalized when the interval $[-\pi, \pi]$ is replaced by $[-L, L]$ for any $L > 0$?
- If f is not defined outside of the interval $[-\pi, \pi]$ (or $[-L, L]$ as the case may be), what is the behavior of the Fourier series for x outside of this interval?
- If f has one or more finite jump discontinuities, what does the series converge to at these points? (convergence in the mean)