Aug. 14 Math 2253H sec. 05H Fall 2014

Section 1.4: The Tangent & Velocity Problems

Motivational Example: Suppose a ball is dropped from the top of the Space Needle 605 feet high. According to Galileo's law, the distance $s(t)$ feet the ball has fallen after t seconds is (neglecting wind drag)

$$s(t) = 16t^2.$$

We define **average velocity** as

$$\text{change in position} \quad \div \quad \text{change in time.}$$

average velocity = change in position $\div$ change in time

Find the average velocity over the period from $t = 0$ to $t = 2$.

positions: $S(2) = 16(2^2) = 16 \cdot 4 = 64$ feet

$S(0) = 16(0^2) = 16 \cdot 0 = 0$ ft

$$\text{Avg vel} = \frac{S(2) - S(0)}{2 - 0} = \frac{64 - 0}{2} = 32 \ \frac{\text{ft}}{\text{sec}}$$

average velocity = change in position $\div$ change in time

Find the average velocity over the period from $t = 2$ to $t = 4$.

positions: $s(4) = 16(4^2) = 16 \cdot 16 = 256$

$s(2) = 64$

$$\text{Avg. vel} = \frac{s(4) - s(2)}{4-2} = \frac{256-64}{2} = \frac{192}{2} = 96\ \frac{\text{ft}}{\text{sec.}}$$

looks like $\dfrac{\Delta s}{\Delta t}$ Δ - denotes change in

Here's a tougher question...

What is the *instantaneous velocity* when $t = 2$?

The expression $\dfrac{s(2) - s(2)}{2-2}$ is nonsense.

Try letting $t = 2$ and $t = 2 + h$ where $h \neq 0$ (but it's small)

$$\text{Avg vel} = \frac{s(2+h) - s(2)}{2+h-2} = \frac{s(2+h) - s(2)}{h}$$

Estimating instantaneous velocity using intervals of decreasing size...

h	$\frac{s(2+h)-s(2)}{h}$	h	$\frac{s(2+h)-s(2)}{h}$
1	80	-1	48
0.1	65.6	-0.1	62.4
0.05	64.8	-0.05	63.2
0.01	64.16	-0.01	63.84

Do the avg. velocities appear to be approach one well defined value?

Perhaps the instantaneous velocity is

$$64 \ \frac{\text{ft}}{\text{sec.}}$$

Slope

If we consider the independent variable t and dependent variable $y = s(t)$, we note that the velocity has the form

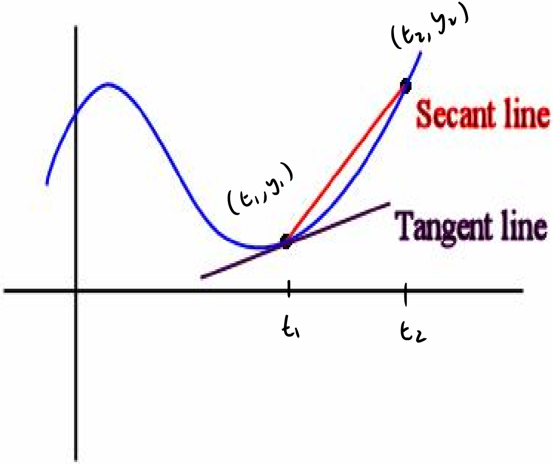
$$\frac{\text{change in } y}{\text{change in } t} = \frac{\text{rise}}{\text{run}} = \text{slope.}$$

The question is, slope of what?

Graphical Interpretations

$y_1 = s(t_1)$

$y_2 = s(t_2)$



Slope

$$\frac{y_2 - y_1}{t_2 - t_1}$$

Figure: A secant line and a tangent line to the graph of a function.

Definitions

Given a graph of a function $y = f(x)$:

A **secant** line is a line connecting two points $P = (x_0, y_0)$ and $Q = (x_1, y_1)$ on the graph. The slope of a secant line is

$$\frac{\Delta y}{\Delta x} = \frac{y_1 - y_0}{x_1 - x_0} = \frac{f(x_1) - f(x_0)}{x_1 - x_0}.$$

A **tangent** line to a curve is a line that "just touches" the curve at a point and has the same direction (a.k.a. slope or rate of change) as the curve at that point. The slope of a tangent line will require a bit of careful consideration. We'll define it as a **limit** of slopes of secant lines.

Example: $y = x^2$ Find the equation of the secant line through

(a) $(1,1)$ and $(2,4)$

$$\frac{\Delta y}{\Delta x} = \frac{4-1}{2-1} = 3$$

Eqn: $y - 1 = 3(x-1)$

$$y = 3x - 2$$

(b) $(1, 1)$ and $(0, 0)$

$$\frac{\Delta y}{\Delta x} = \frac{0-1}{0-1} = 1$$

Eqn of secant line: $y = x$

Find the equation of the line tangent to the graph of $y = x^2$ at $(1, 1)$.

Find the slope by doing the following:

Let $P = (1, 1)$, and set $Q = (1 + h, (1 + h)^2)$ where h is a *small, nonzero* number. Try to deduce the slope of the tangent line by taking h smaller and smaller.

Slope of secant line $\quad \frac{\Delta y}{\Delta x} = \frac{(1+h)^2 - 1^2}{1+h-1}$

$$= \frac{1+2h+h^2-1}{h} = \frac{2h+h^2}{h}$$

$$\frac{\Delta y}{\Delta x} = \frac{\cancel{h}(2+h)}{\cancel{h}} \Rightarrow \frac{\Delta y}{\Delta x} = 2+h$$

for h "very small" $\quad \frac{\Delta y}{\Delta x} \approx 2$

Our slope will be taken to be $m = 2$

The line has eqn:

$$y - 1 = 2(x-1)$$

$$\boxed{y = 2x - 1}$$