Aug. 14 Math 2253H sec. 05H Fall 2014

Section 1.5: Limits

Definition: Let $f(x)$ be defined on an open interval containing the point a except possibly at a. We write

$$\lim_{x\to a} f(x) = L$$

and say "the limit as x approaches a of $f(x)$ equals L" provided we can make f arbitrarily close to L by taking x sufficiently close to a but not equal to a.

We'll accept this definition for now and give a precise definition in section 1.7

Example: Estimate the following limit using a calculator[1]

$$\lim_{x\to 2} \frac{x^2-4}{x-2}$$

Let $f(x) = \frac{x^2-4}{x-2}$, then f is well defined for all real $x \neq 2$.

x	$f(x)$
1	3
1.5	3.5
1.9	3.9
1.99	3.99

x	$f(x)$
3	5
2.5	4.5
2.1	4.1
2.01	4.01

$\vdots$

It appears that

$$\lim_{x\to 2} \frac{x^2-4}{x-2} = 4$$

Note $f(2)$ isn't defined!

[1] We'll learn how to do this without technology in the next sections.

Example: Estimate the following limit using a calculator

$$\lim_{t\to 0} \frac{\sqrt{1+t}-1}{t}$$

$f(t) = \frac{\sqrt{1+t}-1}{t}$ has domain $\{t \mid t \geq -1,\ t \neq 0\}$.

$$\lim_{t\to 0} \frac{\sqrt{1+t}-1}{t} = \frac{1}{2}$$

Keeping 3 decimal places:

t	$f(t)$
0.1	0.488
0.01	0.499
0.001	0.500

t	$f(t)$
-0.1	0.513
-0.01	0.501
-0.001	0.500

By TI-84 plus

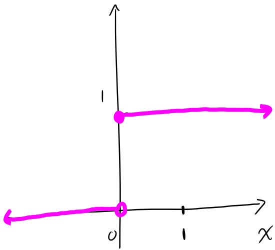
$f(10^{-12}) = 0$

Graphical Limit & A Limit That Doesn't Exist

Consider $H(x) = \begin{cases} 0, & x < 0 \\ 1, & x \geq 0 \end{cases}$. Evaluate if possible

Heaviside Step function

$$\lim_{x\to 1} H(x), \qquad \lim_{x\to 0} H(x)$$



$\lim_{x\to 1} H(x) = 1$

$\lim_{x\to 0} H(x)$ The limit does not exist

we can write

$\lim_{x\to 0} H(x)$ DNE

on the left $\lim_{x\to 0} H(x)$ looks like zero

on the right it looks like 1

One Sided Limits

Definition: We write

$$\lim_{x\to a^-} f(x) = L$$

and say that the left hand limit (or limit from below) of $f(x)$ as x approaches a equals L provided we can make $f(x)$ arbitrarily close to L by taking x sufficiently close to but less than a.

One Sided Limits

Definition: We write
$$\lim_{x\to a^+} f(x) = L$$
and say that the right hand limit (or limit from above) of $f(x)$ as x approaches a equals L provided we can make $f(x)$ arbitrarily close to L by taking x sufficiently close to but larger than a.

Theorem

$$\lim_{x\to a} f(x) = L \quad \text{if and only if} \lim_{x\to a^-} f(x) = L \quad \text{and} \lim_{x\to a^+} f(x) = L$$

Limits Graphically

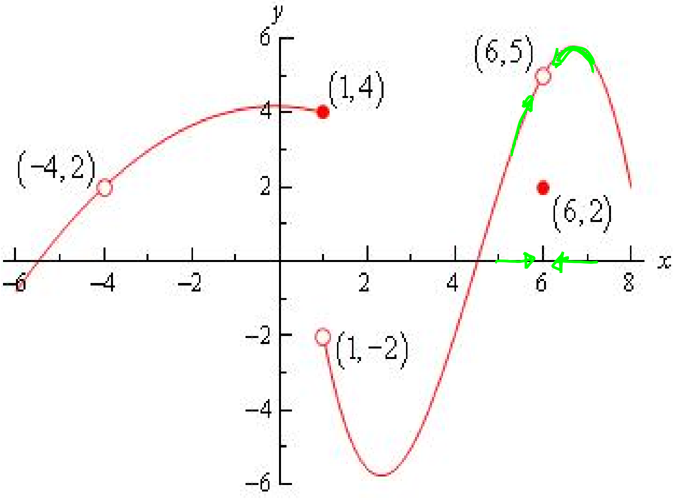


Figure: Graph of a function $y = f(x)$.

Evaluating Limits Using A Graph

Evaluate if possible:

(a) $\lim_{x \to 1^-} f(x) = 4$

(b) $\lim_{x \to 1^+} f(x) = -2$

(c) $\lim_{x \to 1} f(x)$ DNE since $\lim_{x \to 1^-} f(x) \neq \lim_{x \to 1^+} f(x)$

Evaluating Limits Using A Graph

Evaluate if possible:

(d) $\lim_{x \to -4^-} f(x) = 2$

(e) $\lim_{x \to -4^+} f(x) = 2$

(f) $\lim_{x \to -4} f(x) = 2$

Evaluating Limits Using A Graph

Evaluate if possible:

(g) $\lim_{x \to 6} f(x) = 5$

(h) $f(6) = 2$

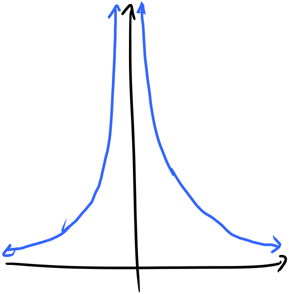
(i) $f(1) = 4$

Infinite Limits

Evaluate

$$\lim_{x \to 0} \frac{1}{x^2}$$

$y = \frac{1}{x^2}$



The y-values become unboundedly large as $x \to 0$ from either side.

We'll write

$$\lim_{x \to 0} \frac{1}{x^2} = \infty$$

(or $+\infty$)

Infinite Limits

Definition: Let $f(x)$ be defined on an open interval containing a except possibly at a. Then

$$\lim_{x\to a} f(x) = \infty$$

provided $f(x)$ can be made arbitrarily large by taking x sufficiently close to a. (The definition of

$$\lim_{x\to a} f(x) = -\infty$$

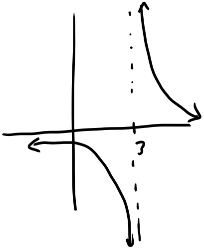
is similar except that f can be made arbitrarily large and negative.)

Again, we will make precise terms like *arbitrary* and *sufficient* in section 1.7

Evaluate if possible

$$\lim_{x\to 3^+} \frac{2}{x-3} = \infty$$

$$\lim_{t\to \frac{\pi}{2}^-} \tan(t) = \infty$$

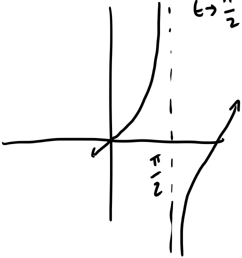


$$\lim_{x\to 3^-} \frac{2}{x-3} = -\infty$$

$$\lim_{x\to 3} \frac{2}{x-3} \quad \text{DNE}$$

$$\lim_{t\to \frac{\pi}{2}} \tan t$$

DNE



The line $x = a$ is a vertical asymptote to the graph of f if

$$\lim_{x\to a^-} f(x) = \pm\infty, \quad \text{or} \quad \lim_{x\to a^+} f(x) = \pm\infty$$