Aug. 18 Math 2253H sec. 05H Fall 2014

Section 1.6 Evaluating Limits: Limit Laws

Suppose

$$\lim_{x\to a} f(x) = L, \quad \lim_{x\to a} g(x) = M, \quad \text{and } c \text{ is constant.}$$

$$\lim_{x\to a}(f(x)\pm g(x)) = L\pm M$$

$$\lim_{x\to a} cf(x) = cL$$

$$\lim_{x\to a} f(x)g(x) = LM$$

$$\lim_{x\to a} \frac{f(x)}{g(x)} = \frac{L}{M} \quad \text{if} \quad M \neq 0$$

Example: Using Limit Laws

$$\lim_{x \to a} c = c \quad \text{and} \quad \lim_{x \to a} x = a$$

Evaluate

$$\lim_{x \to 2} (x^2 + 3x) = \lim_{x \to 2} x^2 + \lim_{x \to 2} 3x$$

$$= \lim_{x \to 2} x \cdot x + 3 \lim_{x \to 2} x$$

$$= \left(\lim_{x \to 2} x\right)\left(\lim_{x \to 2} x\right) + 3 \lim_{x \to 2} x$$

$$= 2(2) + 3(2) = 4 + 6 = 10$$

Example: Using Limit Laws

$$\lim_{x\to a} c = c \quad \text{and} \quad \lim_{x\to a} x = a$$

Evaluate

$$\lim_{x\to -3} \frac{3x-2}{x+5}$$

$$= \frac{\lim_{x\to -3} 3x-2}{\lim_{x\to -3} x+5} = \frac{-11}{2}$$

let $g(x) = x+5$

$$\lim_{x\to -3} g(x) = \lim_{x\to -3} x+5$$

$$= \lim_{x\to -3} x + \lim_{x\to -3} 5$$

$$= -3 + 5 = 2$$

$* \; 2 \neq 0$

More limit laws:

$$\lim_{x \to a} (f(x))^n = L^n$$

$$\lim_{x \to a} \sqrt[n]{f(x)} = \sqrt[n]{L} \quad \text{(if this is defined)}$$

If $f(x)$ is a polynomial or a rational function, and a is in the domain of f, then

$$\lim_{x \to a} f(x) = f(a).$$

Evaluate

$$\lim_{x \to 8} \sqrt[3]{x}$$

$$= \sqrt[3]{\lim_{x \to 8} x}$$

$$= \sqrt[3]{8} = 2$$

Recall that

$$\lim_{x \to 8} x = 8$$

Evaluate

$$\lim_{u\to 4} \frac{u+1}{u^3-2u^2}$$

$$= \frac{4+1}{4^3-2\cdot 4^2} = \frac{5}{32}$$

Let $f(u) = \dfrac{u+1}{u^3-2u^2}$

Is 4 in the domain of f?

$$f(4) = \frac{4+1}{4^3-2\cdot 4^2} = \frac{5}{64-32}$$

$$= \frac{5}{32}$$

which is well defined.

So 4 is in the domain.

More Exotic Limits: Evaluate

Theorem: If $f(x) = g(x)$ for all x on an interval containing a, except possibly at a, then

$$\lim_{x\to a} f(x) = \lim_{x\to a} g(x)$$

provided this limit exists.

This means that we can use some algebraic techniques to evaluate limits. If f is too tough to handle, perhaps we can find a function g that is somehow more manageable.

More Exotic Limits: Evaluate

$$\lim_{x\to 2} \frac{x^2-4}{x-2}$$

$$= \lim_{x\to 2} \frac{(x-\cancel{2})(x+2)}{x-\cancel{2}}$$

$$= \lim_{x\to 2} x+2$$

$$= 2+2 = 4$$

Let $f(x) = \frac{x^2-4}{x-2} = \frac{(x-2)(x+2)}{x-2}$

for $x \neq 2$

$$\frac{(x-2)(x+2)}{x-2} = x+2$$

If $g(x) = x+2$ then

$f(x) = g(x)$ for all $x \neq 2$

$$\lim_{t\to 0} \frac{\sqrt{t+1}-1}{t}$$

$\sqrt{t+1}-1$ goes to zero as $t\to 0$

To get rid of the radical, we rationalize.

The conjugate of $\sqrt{t+1}-1$ is $\sqrt{t+1}+1$

$$\frac{\sqrt{t+1}-1}{t} = \left(\frac{\sqrt{t+1}-1}{t}\right)\cdot\left(\frac{\sqrt{t+1}+1}{\sqrt{t+1}+1}\right)$$

$$= \frac{t+1-1}{t(\sqrt{t+1}+1)} = \frac{t}{t(\sqrt{t+1}+1)}$$

$$= \frac{1}{\sqrt{t+1}+1} \quad \text{for } t\neq 0$$

We have $f(t) = \dfrac{\sqrt{t+1} - 1}{t} = g(t) = \dfrac{1}{\sqrt{t+1} + 1}$

for all $t \neq 0$ $(t \geqslant -1)$

$$\lim_{t \to 0} \frac{\sqrt{t+1} - 1}{t} = \lim_{t \to 0} \frac{1}{\sqrt{t+1} + 1} = \frac{1}{1+1}$$

$$= \frac{1}{2}$$

Evaluate

$$\lim_{r\to 0}\left(\frac{1}{r}-\frac{1}{r^2+r}\right) = \lim_{r\to 0}\frac{1}{r} - \lim_{r\to 0}\frac{1}{r^2+r}$$

$$= \frac{\lim_{r\to 0} 1}{\lim_{r\to 0} r} - \frac{\lim_{r\to 0} 1}{\lim_{r\to 0} r^2+r}$$

unfortunately, both are undefined.

Try writing the difference as one fraction:

$$\lim_{r\to 0}\left(\frac{1}{r}-\frac{1}{r^2+r}\right) = \lim_{r\to 0}\left(\frac{1}{r}-\frac{1}{r(r+1)}\right)$$

$$= \lim_{r\to 0} \left(\frac{r+1}{r(r+1)} - \frac{1}{r(r+1)} \right)$$

$$= \lim_{r\to 0} \frac{r+1-1}{r(r+1)} = \lim_{r\to 0} \frac{r}{r(r+1)}$$

$$= \lim_{r\to 0} \frac{1}{r+1} = \frac{1}{0+1} = 1$$

$*$ Please don't write $\lim_{r\to 0} = 1$ $*$

Perhaps we need more...

$$\lim_{\theta \to 0} \theta^2 \sin \frac{1}{\theta}$$

No method thus far will facilitate taking this limit.

Squeeze Theorem:

Theorem: Suppose $f(x) \leq g(x) \leq h(x)$ for all x in an interval containing a except possibly at a. If

$$\lim_{x\to a} f(x) = \lim_{x\to a} h(x) = L$$

then

$$\lim_{x\to a} g(x) = L.$$

Squeeze Theorem:

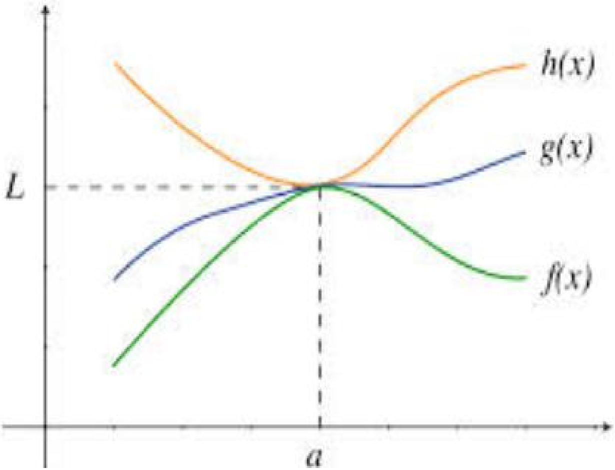

Figure: Graphical Representation of the Squeeze Theorem.

Example: Evaluate

$$\lim_{\theta\to 0} \theta^2 \sin\frac{1}{\theta}$$

Let $g(\theta) = \theta^2 \sin\frac{1}{\theta}$.

Recall

$$\left|\sin\frac{1}{\theta}\right| \leq 1$$

i.e. $-1 \leq \sin\frac{1}{\theta} \leq 1$

Mult by θ^2

$$-\theta^2 \leq \theta^2 \sin\frac{1}{\theta} \leq \theta^2$$

$$\lim_{\theta \to 0} -\theta^2 = 0 \qquad \lim_{\theta \to 0} \theta^2 = 0$$

$$-\theta^2 = -(\theta^2) \neq (-\theta)^2$$

By the squeeze theorem

$$\lim_{\theta \to 0} \theta^2 \sin \frac{1}{\theta} = 0$$