Aug. 19 Math 2253H sec. 05H Fall 2014

Sec. 1.6: A Limit Taking Overview

If f(x) is made up of polynomials, roots, products, quotients, and/or trigonometric functions, try evaluating the limit by substitution (i.e. evaluate f(a) if possible). If this fails, try using factoring, rationalizing, algebraic manipulation or trig. IDs.

Evaluate

$$\lim_{x\to 3} \sqrt{x^3 - 4x} = \sqrt{27-12} = \sqrt{15}$$

Evaluate

if $t=4$ $\quad t^2-4t=16-16=0$

$t^2-3t-4=16-12-4=0$

$$\lim_{t\to 4}\frac{t^2-4t}{t^2-3t-4}$$

$$=\lim_{t\to 4}\frac{t(t-4)}{(t-4)(t+1)}$$

$$=\lim_{t\to 4}\frac{t}{t+1}=\frac{4}{4+1}=\frac{4}{5}$$

Evaluate

$$\lim_{x\to 2} \frac{x-2}{\sqrt{4x+1}-3} = \lim_{x\to 2} \left(\frac{x-2}{\sqrt{4x+1}-3}\right)\cdot\left(\frac{\sqrt{4x+1}+3}{\sqrt{4x+1}+3}\right)$$

$$= \lim_{x\to 2} \frac{(x-2)(\sqrt{4x+1}+3)}{4x+1-9}$$

$$= \lim_{x\to 2} \frac{(x-2)(\sqrt{4x+1}+3)}{4x-8}$$

$$= \lim_{x\to 2} \frac{(x-2)(\sqrt{4x+1}+3)}{4(x-2)} = \lim_{x\to 2} \frac{\sqrt{4x+1}+3}{4} = \frac{6}{4} = \frac{3}{2}$$

Evaluate

$$\lim_{u\to 2^+} \frac{|u-2|}{2-u}$$

$$= \lim_{u\to 2^+} \frac{u-2}{2-u}$$

$$= \lim_{u\to 2^+} \frac{-(2-u)}{2-u}$$

$$= \lim_{u\to 2^+} \frac{-1}{1} = -1$$

$$* \ |x| = \begin{cases} x, & x \geq 0 \\ -x, & x < 0 \end{cases}$$

$$|u-2| = \begin{cases} u-2, & u-2 \geq 0 \\ -(u-2), & u-2 < 0 \end{cases}$$

$u-2 \geq 0 \Rightarrow u \geq 2$

$u-2 < 0 \Rightarrow u < 2$

Section 1.7: Precise Definition of a Limit

Recall the definition we gave back in section 1.5:

Definition: Let $f(x)$ be defined on an open interval containing the point a except possibly at a. We write

$$\lim_{x\to a} f(x) = L$$

and say "the limit as x approaches a of $f(x)$ equals L" provided we can make f arbitrarily close to L by taking x sufficiently close to a.

The words "arbitrary" and "sufficient" are not very precise. We can actually quantify these notions.

Motivating Example

Consider the function $f(x) = \begin{cases} 2x - 1, & x \neq 3 \\ 6, & x = 3 \end{cases}$

$f(3) = 6$

$\lim_{x \to 3} f(x) = 5$

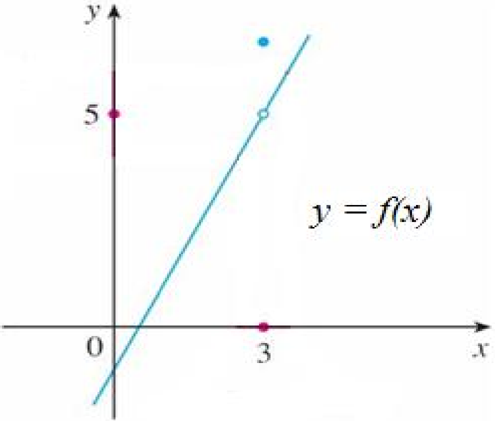


Figure: From the graph, it seems clear that $\lim_{x\to 3} f(x) = 5$.

Our Example: $f(x) = \begin{cases} 2x - 1, & x \neq 3 \\ 6, & x = 3 \end{cases}$

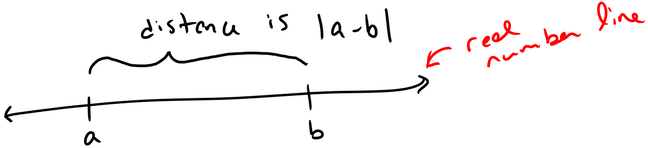
We consider the statement

$$\lim_{x \to 3} f(x) = 5.$$

Question: How close to 3 must x be (keeping $x \neq 3$) so that $f(x)$ is within 0.1 *units* of 5?

Closeness of two numbers

Definition: The distance between a pair of numbers a and b is the absolute value of their difference: $|a - b|$.



So to say $f(x)$ should be within 0.1 units of 5

we can specify

$$|f(x) - 5| < 0.1$$

To say $x \neq 3$ we can specify that

$$|x-3| \neq 0$$

which is equivalent to saying $|x-3| > 0$.

Recall that $|a| < b$ implies

$$-b < a < b \quad \text{for } b > 0.$$

For example $|f(x)-5| < 0.1 \Rightarrow -0.1 < f(x)-5 < 0.1$

Our Question Restated:

Can we find a positive number δ such that

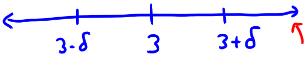
$$|f(x) - 5| < 0.1 \quad \text{if} \quad 0 < |x - 3| < \delta?$$

Note: $|x-3| < \delta \implies -\delta < x-3 < \delta \implies$

$3-\delta < x < 3+\delta$

x is in here

$x \neq 3$



number line

$f(x) = 2x-1$ if $x \neq 3$

We'll find δ:

$|f(x) - 5| < 0.1 \implies |2x-1-5| < 0.1$

$$\Rightarrow \quad |2x-6| < 0.1 \quad \Rightarrow \quad |2(x-3)| < 0.1$$

$$\Rightarrow \quad |2|\,|x-3| < 0.1$$

i.e. $\quad 2|x-3| < 0.1$

$$|x-3| < \frac{0.1}{2} = 0.05$$

It appears that if $\quad 0 < |x-3| < 0.05$, then we're guaranteed that

$$|f(x)-5| < 0.1$$

Set $\quad \delta = 0.05$

What if we want f even closer?

Can we find a positive number δ such that

$$|f(x) - 5| < 0.01 \quad \text{if} \quad 0 < |x - 3| < \delta?$$

$|f(x) - 5| < 0.01 \implies |2x - 1 - 5| < 0.01$

$2|x - 3| < 0.01 \implies |x - 3| < \frac{0.01}{2} = 0.005$

This would need $\delta = 0.005$

Closer still?

Can we find a positive number δ such that

$$|f(x) - 5| < 0.001 \quad \text{if} \quad 0 < |x - 3| < \delta?$$

We'll need $2|x-3| < 0.001$

giving $\delta = \frac{0.001}{2} = 0.0005$.

Arbitrarily Close?

If ϵ is any positive number, can we find a positive number δ such that

$$|f(x) - 5| < \epsilon \quad \text{if} \quad 0 < |x - 3| < \delta?$$

The computations don't change, we get

$$2|x-3| < \varepsilon \implies |x-3| < \frac{\varepsilon}{2}$$

as our condition.

We can set $\delta = \frac{\varepsilon}{2}$.

Graphical Interpretation $f(x) = \begin{cases} 2x - 1, & x \neq 3 \\ 6, & x = 3 \end{cases}$

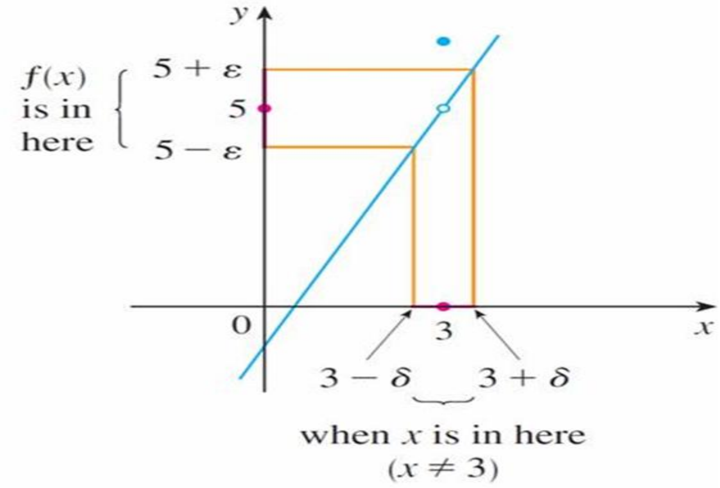


Figure: $|f(x) - 5| < \epsilon$ if $0 < |x - 3| < \delta$.

Precise Definition of a Limit

Definition: Let f be defined on an open interval containing the number a except possibly at a. We write

$$\lim_{x \to a} f(x) = L$$

and say "the limit as x approaches a of $f(x)$ equals L" provided for every $\epsilon > 0$, there exists a number $\delta > 0$ such that

$$\text{if} \quad 0 < |x - a| < \delta \quad \text{then} \quad |f(x) - L| < \epsilon$$

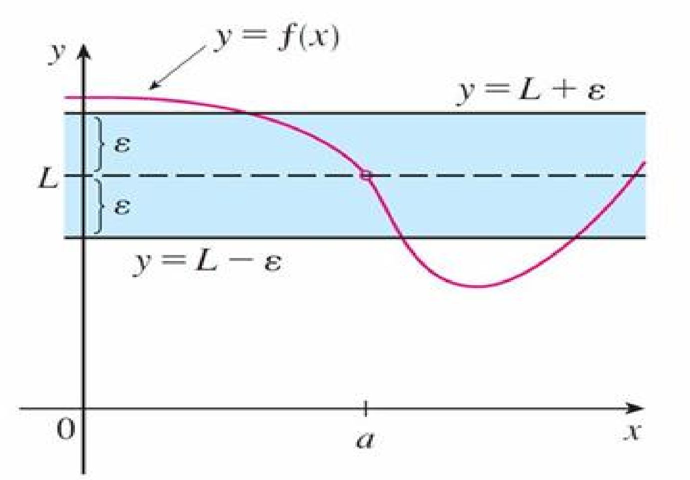


Figure: Graphically, the part of the curve $y = f(x)$ such that $|f(x) - L| < \epsilon$ lives in a horizontal strip. $L - \epsilon < f(x) < L + \epsilon$

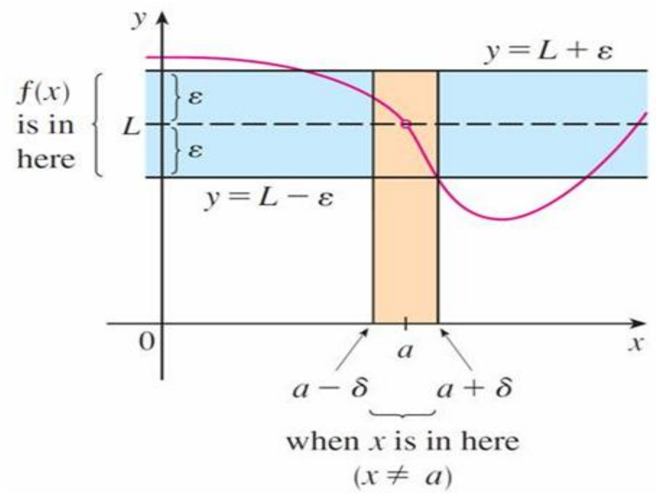


Figure: The numbers x such that $|x - a| < \delta$ would have $y = f(x)$ values that live in a vertical strip.

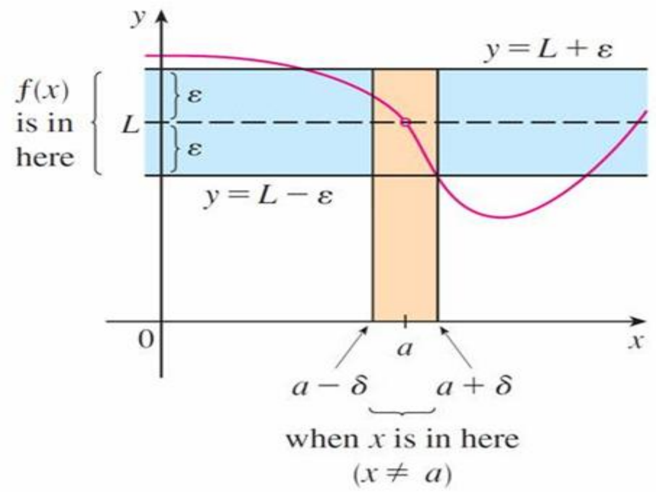


Figure: If the limit of $f(x)$ really is L, then starting with any horizontal strip we'll be able to find a vertical one so that the curve is completely inside the intersection.

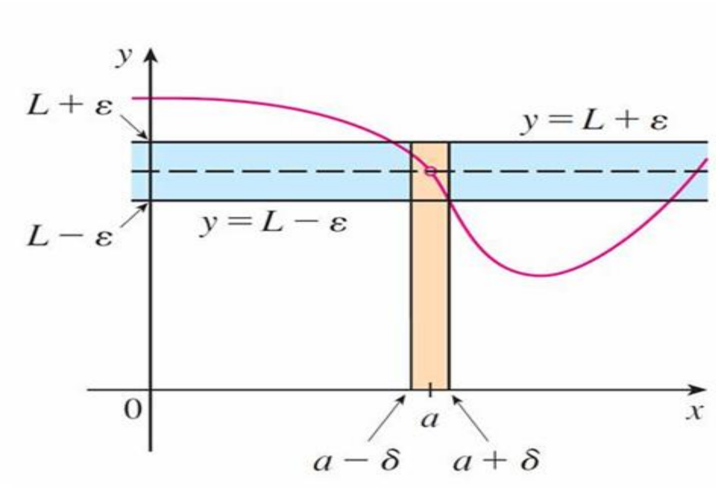


Figure: A smaller ϵ may require a smaller δ. So often, the value of δ depends on ϵ.