Aug. 21 Math 2253H sec. 05H Fall 2014

Section 1.7: Precise Definition of a Limit

Definition: Let f be defined on an open interval containing the number a except possibly at a. We write

$$\lim_{x\to a} f(x) = L$$

and say "the limit as x approaches a of $f(x)$ equals L" provided for every $\epsilon > 0$, there exists a number $\delta > 0$ such that

$$\text{if} \quad 0 < |x-a| < \delta \quad \text{then} \quad |f(x) - L| < \epsilon$$

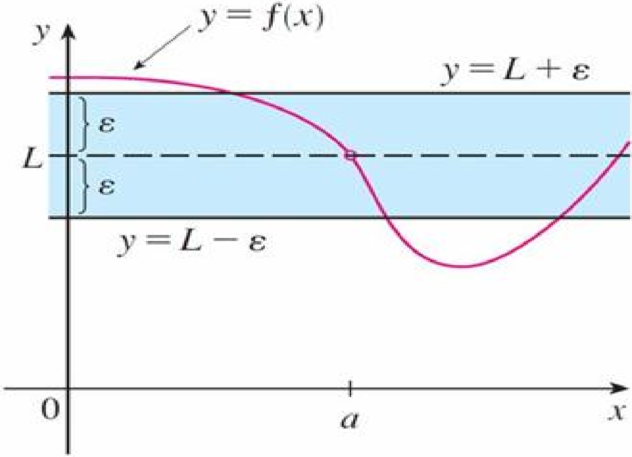


Figure: Graphically, the part of the curve $y = f(x)$ such that $|f(x) - L| < \epsilon$ lives in a horizontal strip. $L - \epsilon < f(x) < L + \epsilon$

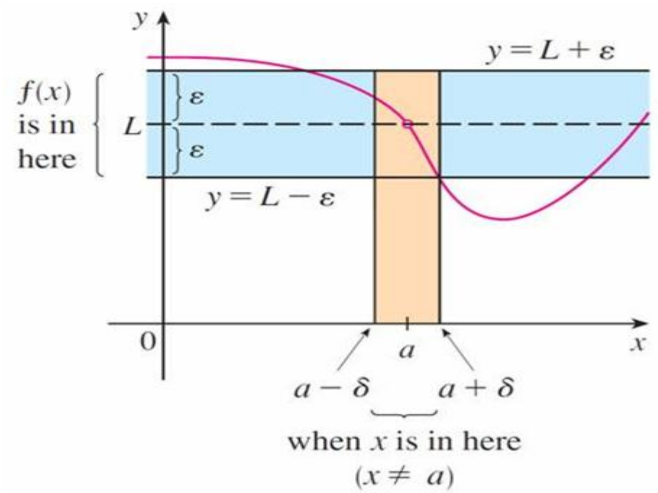


Figure: The numbers x such that $|x - a| < \delta$ would have $y = f(x)$ values that live in a vertical strip.

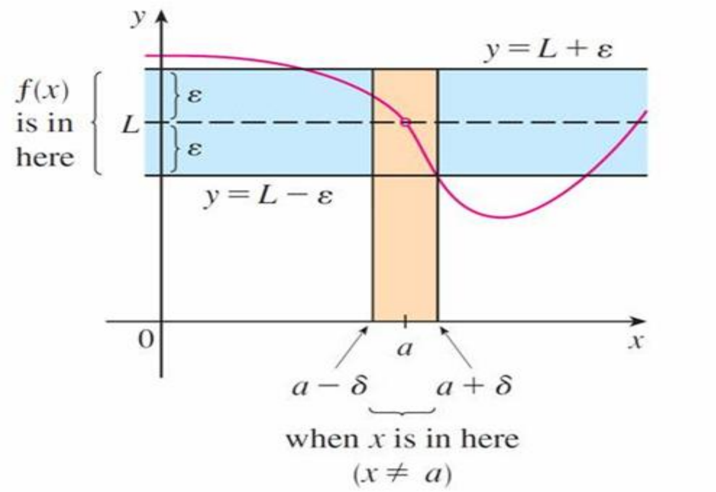


Figure: If the limit of $f(x)$ really is L, then starting with any horizontal strip we'll be able to find a vertical one so that the curve is completely inside the intersection.

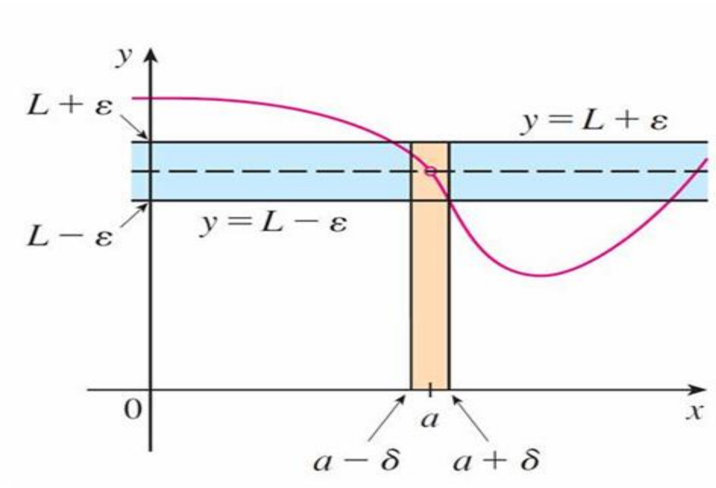


Figure: A smaller ϵ may require a smaller δ. So often, the value of δ depends on ϵ.

Example

Use the formal (i.e. the ϵ-δ) definition of the limit to prove the limit statement

$$\lim_{x \to -3} \frac{x-5}{2} = -4$$

2 phases: ① scratch work to find what δ should be, and ② the actual formal proof.

Scratch work: $f(x) = \frac{x-5}{2}$, $a = -3$, $L = -4$.

we need $|f(x) - (-4)| < \varepsilon$ whenever $|x - (-3)| < \delta$

$$|f(x)-(-4)| = \left|\frac{x-5}{2}+4\right| = \left|\frac{1}{2}(x-5+8)\right| = \left|\frac{1}{2}(x+3)\right|$$

$$= \frac{1}{2}|x+3|$$

So $\quad |f(x)-L| < \varepsilon \implies \frac{1}{2}|x+3| < \varepsilon$

$$\implies |x+3| < 2\varepsilon$$

It appears that $|f(x)-(-4)| < \varepsilon$ is equivalent

to $\quad |x+3| < 2\varepsilon$

So we will take $\delta = 2\varepsilon$.

The formal proof of the statement $\displaystyle\lim_{x\to -3} \frac{x-5}{2} = -4$.

Proof: Let $\varepsilon > 0$. Set $\delta = 2\varepsilon$. Then $\delta > 0$.

Moreover, if $0 < |x-(-3)| < \delta$, then

$$|x+3| < \delta = 2\varepsilon.$$

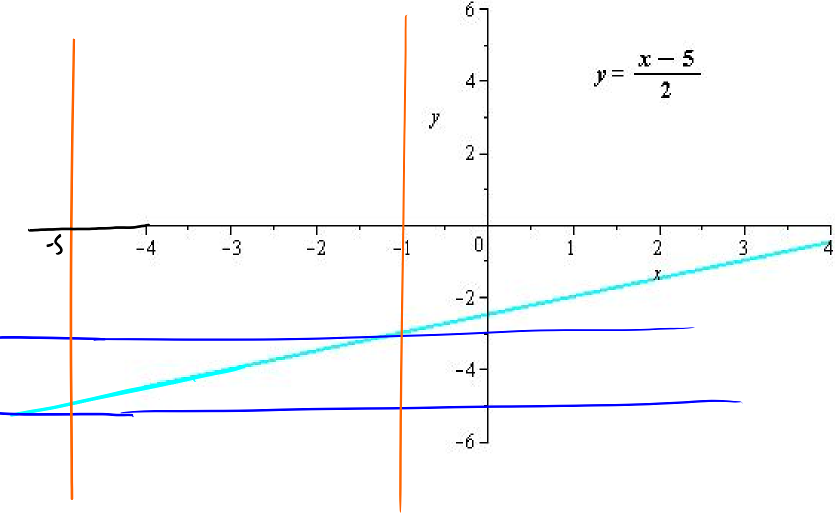
Then $\displaystyle |f(x) - (-4)| = \left|\frac{x-5}{2} + 4\right| = \left|\frac{1}{2}(x-5+8)\right|$

$$= \frac{1}{2}|x+3| < \frac{1}{2}\delta = \frac{1}{2}(2\varepsilon) = \varepsilon .$$

That is $\quad |f(x) - (-4)| < \varepsilon .$

Hence $\displaystyle \lim_{x \to -3} \frac{x-5}{2} = -4$ as required. $\blacksquare$

Graphical Illustration



Example

Use the formal (i.e. the ϵ-δ) definition of the limit to prove the limit statement

$$\lim_{x \to 0} 2x^2 = 0$$

Scratch: $f(x) = 2x^2$, $a = 0$, and $L = 0$

$|f(x) - L| = |2x^2 - 0| = |2x^2| = 2x^2$

We'll need this to be less than ε

$|f(x) - L| < \varepsilon \implies 2x^2 < \varepsilon$

We'll impose $|x-0| < \delta \Rightarrow |x| < \delta$

This gives

$$|x|^2 < \delta^2 \Rightarrow x^2 < \delta^2$$

$$\Rightarrow 2x^2 < 2\delta^2$$

If we set $2\delta^2 = \varepsilon$ we get the required result $2x^2 < \varepsilon$.

We take $\delta = \sqrt{\frac{\varepsilon}{2}}$.

Proof: Let $\varepsilon > 0$. Set $\delta = \sqrt{\frac{\varepsilon}{2}}$. Then

$\delta > 0$. If $0 < |x - 0| < \delta$, then

$|x| < \delta = \sqrt{\frac{\varepsilon}{2}}$. Observe then that

$$|f(x) - 0| = |2x^2 - 0| = |2x^2| = 2x^2$$

$$= 2|x|^2 < 2(\delta)^2 = 2\left(\sqrt{\frac{\varepsilon}{2}}\right)^2 = 2\left(\frac{\varepsilon}{2}\right) = \varepsilon.$$

i.e. $|2x^2 - 0| < \varepsilon$.

This proves that $\lim_{x \to 0} 2x^2 = 0$. $\blacksquare$