Aug. 21 Math 2253H sec. 05H Fall 2014

Section 1.7: Precise Definition of a Limit

Definition: Let f be defined on an open interval containing the number a except possibly at a. We write

$$\lim_{x\to a} f(x) = L$$

and say "the limit as x approaches a of $f(x)$ equals L" provided for every $\epsilon > 0$, there exists a number $\delta > 0$ such that

$$\text{if} \quad 0 < |x-a| < \delta \quad \text{then} \quad |f(x)-L| < \epsilon$$

Example

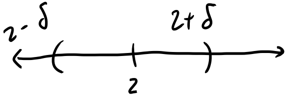
Consider the limit statement

$$\lim_{x \to 2} (x^2 + x - 3) = 3.$$

Use a graph to find a number δ such that if $0 < |x - 2| < \delta$ then

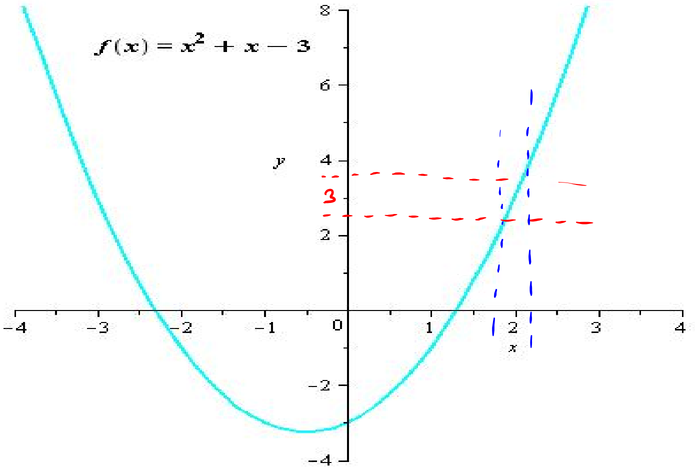
$$|(x^2 + x - 3) - 3| < 0.1.$$

$0 < |x-2| < \delta$

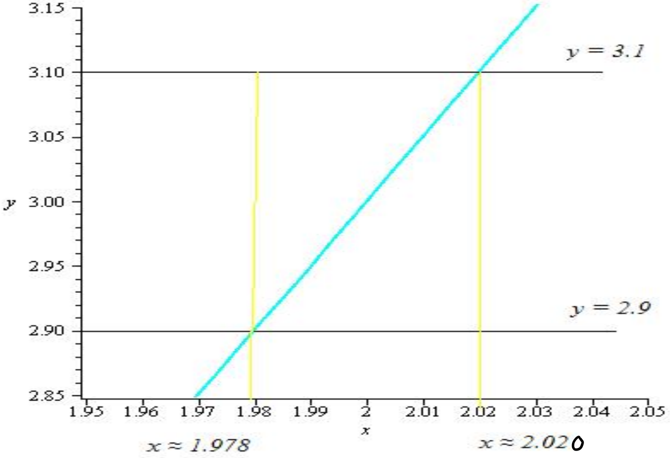


x in here $x \neq 2$

$\lim_{x\to 2}(x^2 + x - 3) = 3.$



$\lim_{x \to 2}(x^2 + x - 3) = 3.$



We could go up to 0.020 units on the right side and up to 0.022 units on the left.

We can take δ to be the smaller $\delta = 0.02$. This guarentees that $f(x)$ is inside the vertical band

$$3 - 0.1 < f(x) < 3 + 0.1$$

Example

Use the formal (i.e. the ϵ-δ) definition of the limit to prove the limit statement

$$\lim_{x\to 2}(x^2+x-3)=3$$

Scratch work: $f(x)=x^2+x-3$, $a=2$, $L=3$

We'll need $|f(x)-3|<\varepsilon$.

We will impose $|x-2|<\delta$.

$|f(x)-3| = |x^2+x-3-3| = |x^2+x-6| = |(x+3)(x-2)|$

$= |x+3|\,|x-2|$

We need to get a bound on $|x+3|$ – i.e. determine its maximum size.

We have $|x-2| < \delta \Rightarrow -\delta < x-2 < \delta$

$2-\delta < x < 2+\delta$ add 3

$5-\delta < x+3 < 5+\delta$

Let insist that $\delta \leq 1$. Then $4 < x+3 < 6$

so then $|x+3| < 6$.

Going back $|f(x)-3| = |x+3|\,|x-2| < 6\delta$

If ① $|x-2| < \delta$ and

② $\delta \le 1$

This motivates taking $6\delta = \varepsilon$ i.e. $\delta = \frac{\varepsilon}{6}$.

We can write $\delta = \min\left\{1, \frac{\varepsilon}{6}\right\}$. This means δ is the smaller of the two. So $\delta \le 1$ and $\delta \le \frac{\varepsilon}{6}$.

Proof: Let $\varepsilon > 0$. Set $\delta = \min\{1, \frac{\varepsilon}{6}\}$. So

$\delta > 0$, $\delta \le 1$, and $\delta \le \frac{\varepsilon}{6}$. Then note

that if $0 < |x-2| < \delta$ then

① $|x-2| < \delta$ and ② $|x+3| < 6$.

So
$$|f(x) - 3| = |x^2 + x - 3 - 3| = |x+3|\,|x-2| < 6\delta$$

$$\le 6\left(\frac{\varepsilon}{6}\right) = \varepsilon.$$

Hence the limit statement is proved. $\blacksquare$