Aug. 25 Math 2253H sec. 05H Fall 2014

Section 1.7: Precise Definition of a Limit

One Sided Limits: Note that $0 < |x - a| < \delta$ means that

$$a - \delta < x < a + \delta \quad \text{and} \quad x \neq a$$

We can say this with the pair of inequalities

$$a < x < a + \delta \quad (x \text{ is to the right of } a)$$

or

$$a - \delta < x < a \quad (x \text{ is to the left of } a)$$

To define one sided limits, we just use one of these two inequalities.

One Sided Limits

Definition: $\lim_{x \to a^+} f(x) = L$

provided for every $\epsilon > 0$ there exists a number $\delta > 0$ such that

$$\text{if} \quad a < x < a + \delta \quad \text{then} \quad |f(x) - L| < \epsilon.$$

Definition: $\lim_{x \to a^-} f(x) = L$

provided for every $\epsilon > 0$ there exists a number $\delta > 0$ such that

$$\text{if} \quad a - \delta < x < a \quad \text{then} \quad |f(x) - L| < \epsilon.$$

Limits at Infinity

We can't really make *an interval of radius ϵ about infinity*. (That is, $|f(x) - \infty| < \epsilon$ doesn't make sense!) So we need a different way of characterizing $f(x) \to \infty$.

We don't try to treat ∞ like a finite number. Rather, we consider $f(x)$ *arbitrarily* large by saying

$$\text{for every} \quad M > 0, \quad f(x) > M.$$

Limits at Infinity

Definition: $\lim_{x \to a} f(x) = \infty$

provided for every $M > 0$ there exists a number $\delta > 0$ such that

$$\text{if} \quad 0 < |x - a| < \delta \quad \text{then} \quad f(x) > M.$$

Definition: $\lim_{x \to a} f(x) = -\infty$

provided for every $M > 0$ there exists a number $\delta > 0$ such that

$$\text{if} \quad 0 < |x - a| < \delta \quad \text{then} \quad f(x) < -M.$$

Example

Prove the limit statement

$$\lim_{x \to 1} \frac{1}{(x-1)^2} = \infty$$

Scratch work: $f(x) = \frac{1}{(x-1)^2}$, $a = 1$, $L = \infty$

Need $f(x) > M$ for any number $M > 0$

We'll impose $|x-1| < \delta$

We need $\dfrac{1}{(x-1)^2} > M$ take the reciprocal

$$\frac{1}{M} > (x-1)^2$$ take square roots

$$\sqrt{\frac{1}{M}} > \sqrt{(x-1)^2} = |x-1|$$

we have $|x-1| < \dfrac{1}{\sqrt{M}}$

This motivates taking $\delta = \dfrac{1}{\sqrt{M}}$.

Proof: Let $M > 0$. Set $\delta = \frac{1}{\sqrt{M}}$. Note $\delta > 0$.

If $0 < |x-1| < \delta$, then

$$|x-1| < \frac{1}{\sqrt{M}} \quad \text{so} \quad (x-1)^2 < \frac{1}{M}$$

and taking reciprocals, we get

$$\frac{1}{(x-1)^2} > M.$$

That is $f(x) > M$. Hence $\lim_{x\to 1} f(x) = \infty$. ∎