Aug. 26 Math 2253H sec. 05H Fall 2014

Section 1.8: Continuity

Definition: A function f is continuous at a number a if

$$\lim_{x \to a} f(x) = f(a).$$

Three implications: ① $f(a)$ exists as a finite number, ② the limit exists as a finite number, ③ the limit and $f(a)$ are the same number.

A function is continuous on an interval I if it is continuous at each point in I.

Formal Definition of Continuity at a Point

Defintion: Let f be defined on an open interval containing the number a. Then f is continuous at a provided for every $\epsilon > 0$ there exists a number $\delta > 0$ such that

$$\text{if} \quad |x - a| < \delta \quad \text{then} \quad |f(x) - f(a)| < \epsilon.$$

Polynomials & Rational Functions

Recall: If f is a polynomial or a rational function, and a is in the domain of f, then

$$\lim_{x\to a} f(x) = f(a).$$

Using the language of **continuity**, this tells us that

Polynomials and Rational Functions are continuous everywhere on their domains.

If a function fails to be continuous at a point, we say that it is **discontinuous** at that point. - it has a discontinuity at that point

Determine where each function is discontinuous.

(a) $f(x) = \dfrac{x^2 - 4x}{x^2 - 3x - 4}$

Since f is rational, its continuous everywhere on its domain.

So we need to find values not in the domain.

$$f(x) = \frac{x(x-4)}{(x+1)(x-4)}$$

the domain is

$$\{x \mid x \neq -1,\ x \neq 4\}$$

The 2 points where f is discontinuous are

-1 and 4.

(b) $f(x) = \begin{cases} 2x, & x < 1 \\ x^2 + 1, & 1 \le x < 2 \\ 3, & x \ge 2 \end{cases}$

The pieces are polynomial hence continuous. We need to check at the points where function changes definition.

$f(1) = 1^2 + 1 = 2$

$\lim_{x\to 1^-} f(x) = \lim_{x \to 1^-} 2x = 2$, $\quad \lim_{x \to 1^+} f(x) = \lim_{x\to 1^+} x^2 + 1 = 2$

$$\lim_{x\to 1} f(x) = f(1)$$

f is continuous @ $\underline{1}$.

Check 2: $f(2) = 3$

$$\lim_{x\to 2^-} f(x) = \lim_{x\to 2^-} x^2+1 = 5$$

$\lim_{x\to 2} f(x)$ can't equal $f(2)$

since one of the limits is 5 and

$f(2) = 3$.

f is discontinuous at 2.

Example:

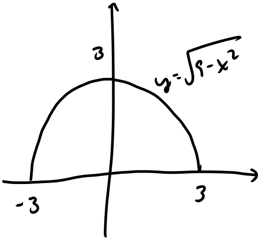
Consider the function $f(x) = \sqrt{9 - x^2}$. Plot a rough sketch of the graph of f, and determine its domain.

Let $y = \sqrt{9-x^2} \implies y^2 = 9 - x^2 \implies x^2 + y^2 = 9$

(top half of circle of radius 3 centered @ $(0,0)$)

Domain: $9 - x^2 \geq 0$

$x^2 \leq 9 \implies |x| \leq 3$



$$\underbrace{\{x \mid -3 \leq x \leq 3\}}_{\text{domain}}$$

$f(x) = \sqrt{9 - x^2}$

Note that f is continuous on $-3 < x < 3$. What can be said about

$$\lim_{x \to -3} f(x) \qquad \text{or} \qquad \lim_{x \to 3} f(x)?$$

These 2 sided limits don't exist because they require f to be defined outside its domain.

Continuity From the Left & Right

Definition: A function f is continuous from the right at a if

$$\lim_{x \to a^+} f(x) = f(a),$$

and is continuous from the left at a if

$$\lim_{x \to a^-} f(x) = f(a).$$

Example: $f(x) = \sqrt{9 - x^2}$

Show that f is continuous from the right at -3 and continuous from the left at 3.

$$f(-3) = \sqrt{9-(-3)^2} = 0$$

$$\lim_{x \to -3^+} f(x) = \lim_{x\to -3^+} \sqrt{9-x^2} = \sqrt{9-9} = 0$$

so $\lim_{x\to -3^+} f(x) = f(-3)$

f is cont. from the right @ -3.

$$f(3) = \sqrt{9-3^2} = 0$$

$$\lim_{x \to 3^-} f(x) = \lim_{x \to 3^-} \sqrt{9-x^2} = 0$$

f is cont. from the left at 3.

A Theorem on Continuous Functions

Theorem If f and g are continuous at a and for any constant c, the following are also continuous at a:

$$(i)\, f+g, \quad (ii)\, f-g, \quad (iii)\, cf, \quad (iv)\, fg, \quad \text{and} \quad (v)\, \frac{f}{g}, \text{ if } g(a) \neq 0.$$

Every polynomial is continuous on $(-\infty, \infty)$, and every rational function is continuous on its domain (everywhere except where the denominator is zero).

Other functions continuous on their domains are **root** functions and **trigonometric** functions.

Evaluate by substitution:

$$\lim_{\theta \to \pi} \frac{1 - \sin\theta}{3 + \cos\theta} = \frac{1 - \sin\pi}{3 + \cos\pi} = \frac{1}{2}$$

$$\lim_{\theta \to \pi} 3 + \cos\theta = 3 + \cos\pi$$
$$= 2$$

$$\lim_{t \to \frac{\pi}{4}} \sec t = \sec\frac{\pi}{4}$$
$$= \sqrt{2}$$

$\frac{\pi}{4}$ is in the domain of the secant function

Find all values of c such that f is continuous on $(-\infty, \infty)$.

$$f(x) = \begin{cases} x + c, & x < 2 \\ cx^2 - 3, & 2 \leq x \end{cases}$$

The two pieces are polynomials, hence everywhere continuous.

We need $\lim_{x \to 2} f(x) = f(2)$.

$f(2) = c(2^2) - 3 = 4c - 3$

$\lim_{x \to 2^-} f(x) = \lim_{x \to 2^-} x + c = 2 + c$

$$\lim_{x\to 2^+} f(x) = \lim_{x\to 2^+} cx^2 - 3 = 4c - 3$$

Continuity requires

$$4c - 3 = 2 + c \implies$$

$$3c = 5 \implies \boxed{c = 5/3}$$

f is continuous on $(-\infty, \infty)$ if and only if $c = \frac{5}{3}$.

Compositions

Suppose $\lim_{x\to a} g(x) = b$, and f is continuous at b, then

$$\lim_{x\to a} f(g(x)) = f(b) \quad \text{i.e.} \quad \lim_{x\to a} f(g(x)) = f\left(\lim_{x\to a} g(x)\right).$$

Theorem: If g is continous at a and f is continuous at $g(a)$, then $(f \circ g)(x)$ is continuous at a.

$$(f \circ g)(x) = f(g(x))$$

Evaluate

$$\lim_{x\to\pi} \cos\left(x + \sin x\right) = \cos(\pi + \sin\pi)$$

$$= \cos(\pi)$$

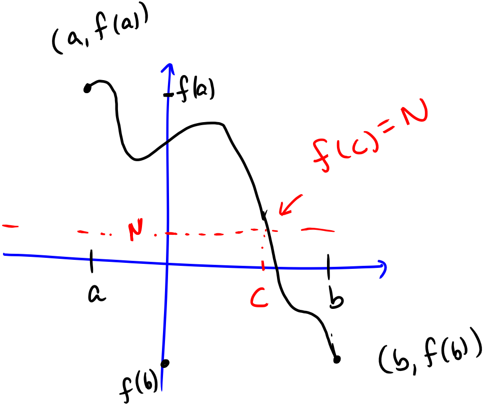
$$= -1$$

$y = x + \sin x$ is continuous on $(-\infty, \infty)$ and

$y = \cos x$ is cont. on $(-\infty, \infty)$.

Theorem:

Intermediate Value Theorem (IVT) Suppose f is continuous on the closed interval $[a, b]$ and let N be any number between $f(a)$ and $f(b)$. Then there exists c in the interval (a, b) such that $f(c) = N$.



If a continuous function takes 2 values, then it must take every value between them!