Nov 7 Math 2253H sec. 05H Fall 2014

Section 5.2: Volumes

We'll call an object a **cylinder** if cross sections taken with respect to some axis are identical.

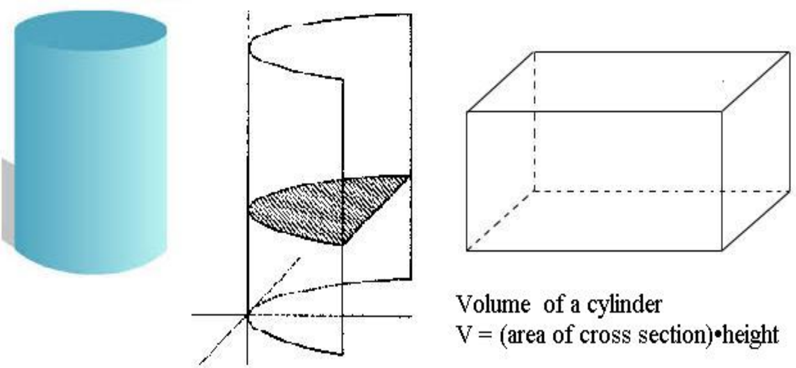


Figure: A circular, a parabolic, and a rectangular cylinder.

Volume of Solid by Slicing

Let the cross sectional area (the yellow shaded) at x_i be $A(x_i)$.

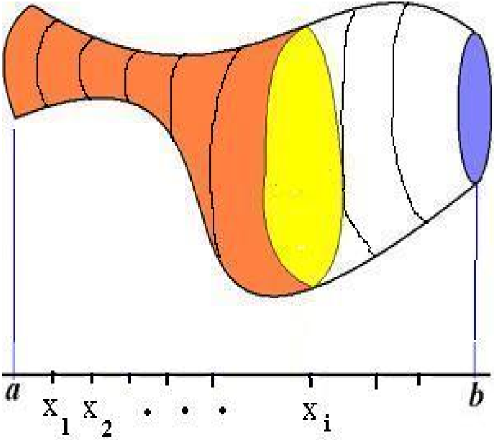


Figure: The total volume $V \approx A(x_1)\Delta x + A(x_2)\Delta x + \cdots + A(x_n)\Delta x$.

Volume of Solid by Slicing

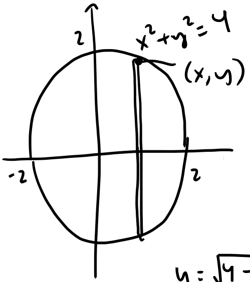
Our volume

$$V \approx A(x_1)\Delta x + A(x_2)\Delta x + \cdots + A(x_n)\Delta x = \sum_{i=1}^{n} A(x_i)\Delta x$$

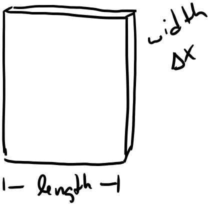
Volume: Let S be a solid that that lies between $x = a$ and $x = b$ having cross sectional area $A(x)$, where the cross section is in the plane through the solid perpendicular to the x-axis at each x in (a, b). The volume of S is

$$V = \lim_{n\to\infty} \sum_{i=1}^{n} A(x_i)\Delta x = \int_a^b A(x)\,dx.$$

An object has as its base the disk $x^2 + y^2 \leq 4$ in the xy-plane. Cross sections taken perpendicular to the x-axis are squares with one side in the plane. Find the volume of the solid.



$y = \sqrt{4-x^2}$



$$\text{length} = 2y$$
$$= 2\sqrt{4-x^2}$$

So area $A(x)$

$$A(x) = \left(2\sqrt{4-x^2}\right)^2$$
$$= 4(4-x^2)$$

for $-2 \leq x \leq 2$

▸ Volume by Cross Section Applet

Volume $V = \int_{-2}^{2} 4(4-x^2)\, dx$

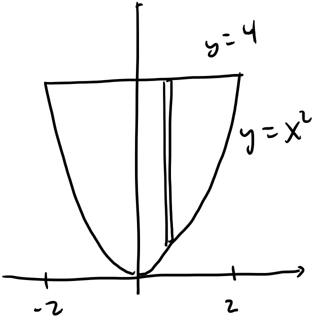
using even symmetry

$$= 2\int_{0}^{2} 4(4-x^2)\, dx$$

$$= 8\left(4x - \frac{x^3}{3}\right)\Big|_0^2$$

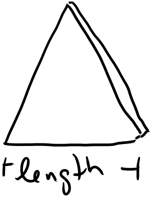
$$= 8\left(8 - \frac{8}{3}\right) = 8\left(\frac{2}{3}\cdot 8\right) = 8\left(\frac{16}{3}\right) = \frac{128}{3}$$

An object has as its base the region bounded between $y = x^2$ and $y = 4$. Cross sections taken perpendicular to the x-axis are equilateral triangles with one side in the plane. Find the volume of the solid.



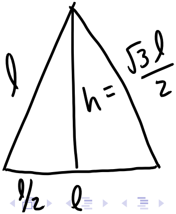
Volume by Cross Section Applet 2

$x^2 = 4 \Rightarrow x = 2$ or $x = -2$



side length is

$4 - x^2$



$$\text{Area} = \frac{1}{2}bh = \frac{1}{2}\ell\left(\frac{\sqrt{3}\ell}{2}\right) = \frac{\sqrt{3}}{4}\ell^2$$

So $A(x) = \frac{\sqrt{3}}{4}(4-x^2)^2$, and the volume

$$V = \int_{-2}^{2} \frac{\sqrt{3}}{4}(4-x^2)^2\,dx$$

Using Symmetry

$$= 2\int_{0}^{2} \frac{\sqrt{3}}{4}(4-x^2)^2\,dx$$

$$= \frac{\sqrt{3}}{2} \int_0^2 (16 - 8x^2 + x^4)\, dx$$

$$= \frac{\sqrt{3}}{2} \left[16x - 8\frac{x^3}{3} + \frac{x^5}{5} \right|_0^2$$

$$= \frac{\sqrt{3}}{2} \left[32 - \frac{64}{3} + \frac{32}{5} \right]$$

$$= 16\sqrt{3}\left(1 - \frac{2}{3} + \frac{1}{5}\right) = 16\sqrt{3}\left(\frac{15 - 10 + 3}{15}\right)$$

$$= \frac{128\sqrt{3}}{15}$$

Solid Formed by Revolution

Consider a region bounded under the nonnegative function $y = f(x)$ for $a \leq x \leq b$. If this region is rotated about the x-axis, a solid is formed. The cross sections of this solid will be circles with radius $f(x)$. So the volume of such a solid is

$$\text{Solid of Revolution:} \quad V = \int_a^b \pi(f(x))^2\, dx$$

This is called the method of **disks.** Each very thin slice is a disk.

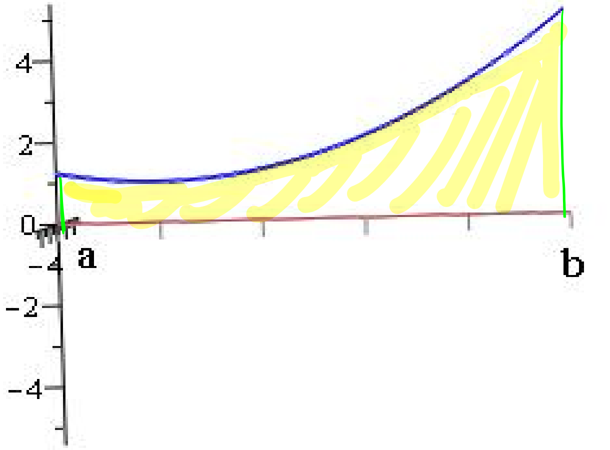


Figure: Start with a positive function $y = f(x)$ and the region below the curve on $[a, b]$

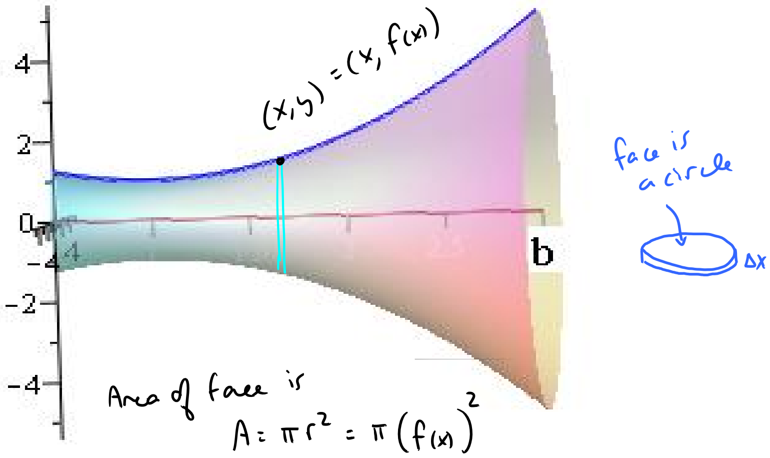


Figure: Revolve it about the x-axis to get a solid whose cross sections are circular disks.

Derive the formula for the volume of a cone $V = \frac{\pi}{3} r^2 h$.

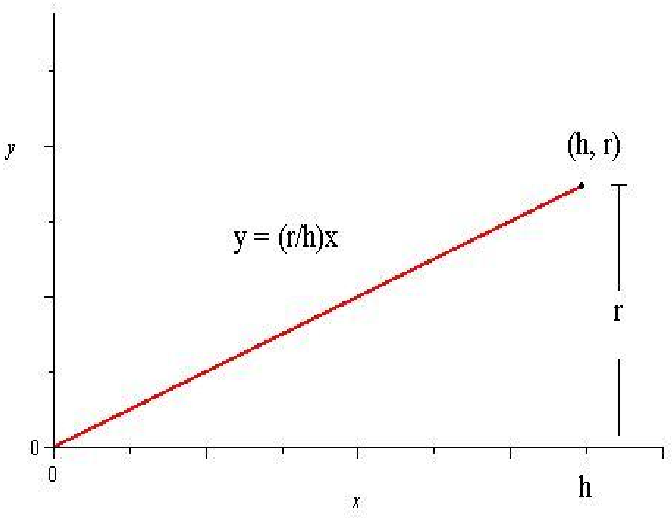


Figure: Start with the line $y = \frac{r}{h} x$ for $0 \le x \le h$.

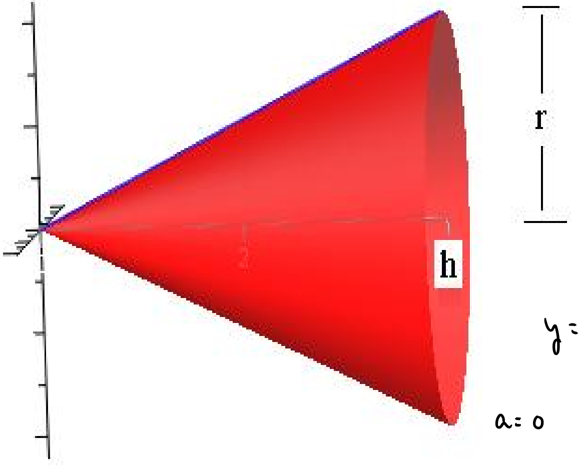


Figure: And revolve this line about the x-axis to get the cone.

The volume

$$V = \int_0^h \pi \left(\frac{r}{h} x\right)^2 dx$$

$$= \int_0^h \pi \frac{r^2}{h^2} x^2 dx = \frac{\pi r^2}{h^2} \int_0^h x^2 dx$$

$$= \frac{\pi r^2}{h^2} \left[\frac{x^3}{3} \right|_0^h$$

$$= \frac{\pi r^2}{h^2} \left[\frac{h^3}{3} - 0 \right] = \frac{\pi r^2 h^3}{3h^2} = \frac{\pi r^2 h}{3}$$