Nov 17 Math 2253H sec. 05H Fall 2014

Section 5.2: Volumes

Solid Formed by Revolution

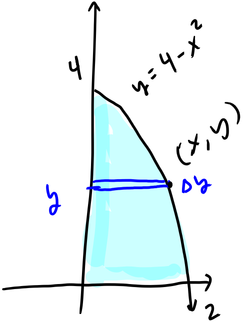
Consider a region bounded under the nonnegative function $y = f(x)$ for $a \leq x \leq b$. If this region is rotated about the x-axis, a solid is formed. The cross sections of this solid will be circles with radius $f(x)$. So the volume of such a solid is

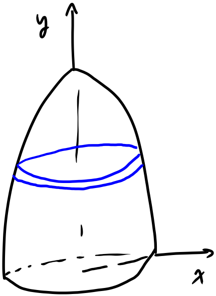
$$\text{Solid of Revolution:} \quad V = \int_a^b \pi (f(x))^2 \, dx$$

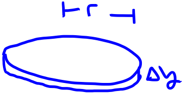
This is called the method of **disks.** Each very thin slice is a disk.

We can construct a similar integral for the volume of a solid of revolution for which the axis of rotation is some other horizontal or vertical line (i.e. not the x-axis)

The first quadrant region bounded between $y = 4 - x^2$ and the x and y axes is rotated about the y-**axis**. Find the volume of the resulting solid.







radius $= x$

$= \sqrt{4-y}$

Volume of one disk

$$y = 4 - x^2 \Rightarrow x^2 = 4 - y \Rightarrow x = \sqrt{4-y}$$

$$V_D = \pi r^2 \Delta y$$
$$= \pi(4-y)\Delta y$$

The total Volume

$$V = \int_0^4 \pi (4-y)\, dy$$

$$= \pi \int_0^4 (4-y)\, dy$$

$$= \pi \left[4y - \frac{y^2}{2} \right|_0^4$$

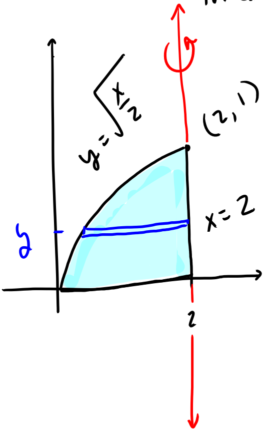
$$= \pi \left[4 \cdot 4 - \frac{4^2}{2} - \left(4 \cdot 0 - \frac{0^2}{2} \right) \right]$$

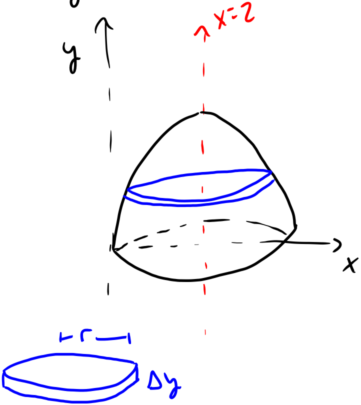
$$= \pi(16-8)$$

$$= 8\pi$$

The region bounded by the curves $y = \sqrt{\frac{x}{2}}$, $y = 0$, and $x = 2$ is rotated about the line $x = 2$. Find the volume of the resulting solid.

Intersection: $y = \sqrt{\frac{2}{2}} = 1$





From

$y = \sqrt{\frac{x}{2}}$

$y^2 = \frac{x}{2}$

$x = 2y^2$

radius $r = 2 - x = 2 - 2y^2$

area is $\pi r^2 = \pi (2 - 2y^2)^2$

The volume of one disk is

$$V_D = \pi (2 - 2y^2)^2 \Delta y$$

The total volume is

$$V = \int_0^1 \pi (2 - 2y^2)^2 \, dy$$

$$= \pi \int_0^1 (4 - 8y^2 + 4y^4)\, dy$$

$$= 4\pi \int_0^1 (1 - 2y^2 + y^4)\, dy$$

$$= 4\pi \left[y - 2\frac{y^3}{3} + \frac{y^5}{5} \right|_0^1$$

$$= 4\pi \left[1 - 2\frac{1^3}{3} + \frac{1^5}{5} - 0 \right]$$

$$= 4\pi\left(1 - \frac{2}{3} + \frac{1}{5}\right)$$

$$= 4\pi\left(\frac{15 - 10 + 3}{15}\right) = \frac{32\pi}{15}$$

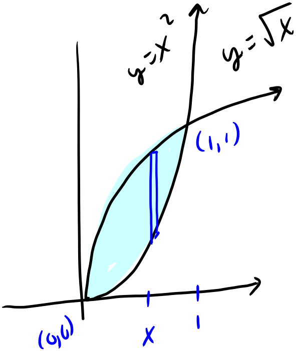
Washers: (solid with a solid part removed)

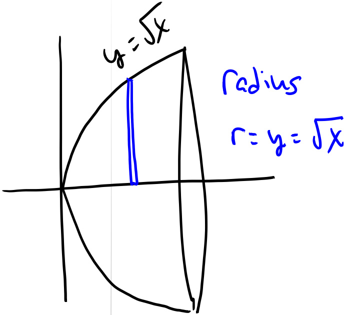
Suppose we consider the region bounded between two curves $y = f(x)$ and $y = g(x)$ for $a \le x \le b$ with $0 \le g(x) \le f(x)$. If this region is rotated about the x-axis, a solid is generated with volume

$$V = \int_a^b \pi((f(x))^2 - (g(x))^2)\, dx$$

This is called the method of **washers.** Each very thin slice is shaped like a washer (a disk with a concentric disk removed).

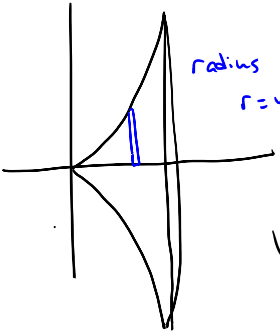
Find the volume of the solid obtained by rotating the region bounded between $y = \sqrt{x}$ and $y = x^2$ about the x-axis.





$A(x) = \pi(\sqrt{x})^2$

▸ Volume by Washers Applet



$A(x) = \pi (x^2)^2$

radius

$r = y = x^2$

Volume $V = \int_0^1 \pi \left((\sqrt{x})^2 - (x^2)^2 \right) dx$

$$= \pi \int_0^1 (x - x^4)\, dx$$

$$= \pi \left[\frac{x^2}{2} - \frac{x^5}{5} \right|_0^1$$

$$= \pi \left[\frac{1^2}{2} - \frac{1^5}{5} - 0 \right]$$

$$= \pi \left(\frac{1}{2} - \frac{1}{5} \right) = \pi \left(\frac{5-2}{10} \right) = \frac{3\pi}{10}$$

Suppose we wish to find the volume of the solid obtained by rotating the first quadrant region bounded by $y = x - x^3$ about the y-axis.

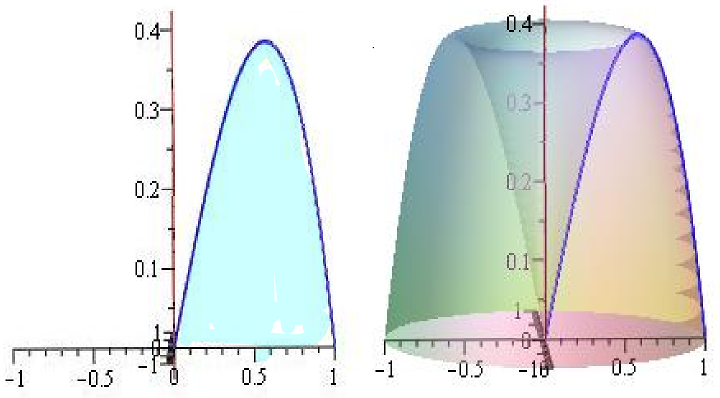


Figure: $y = x - x^3$, and the solid obtained from rotation about the y-axis.

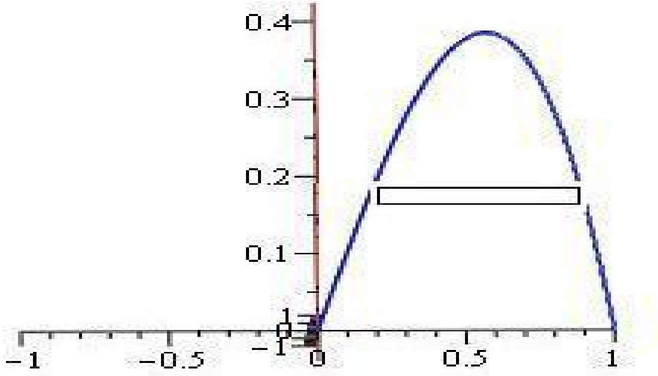


Figure: $y = x - x^3$. We can't find inner and outer radii (left and right functions) because we can't solve for x!!

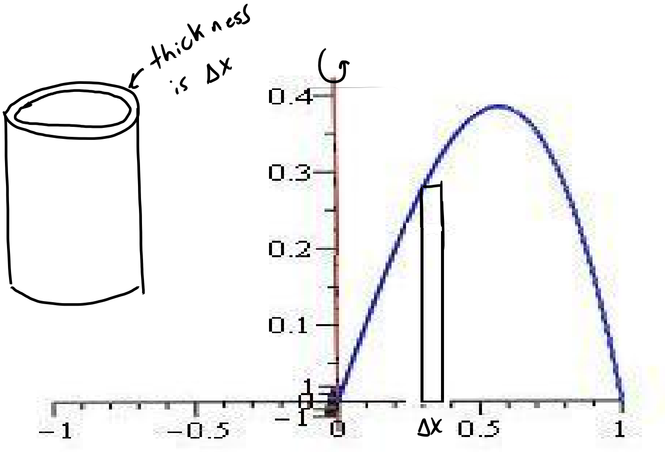


Figure: $y = x - x^3$, We can rotate vertical rectangles, but we won't get disks or washers!!

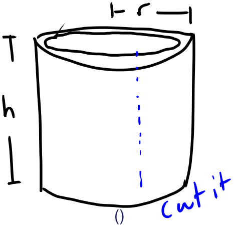
We get cylindrical Shells.

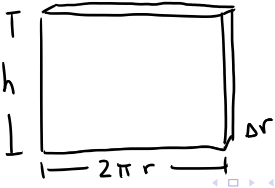
Section 5.3: Method of Cylindrical Shells

Suppose we take a thin strip (with width Δr) that is parallel to the axis of rotation. When we revolve this, we obtain a very thin cylindrical shell with volume

$$V = 2\pi r h \Delta r$$

where r is the average radius (distance between strip and the axis of rotation), and h is the height of the strip.





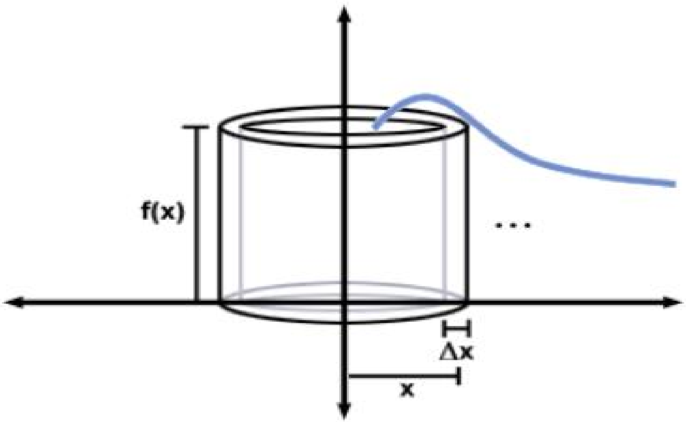


Figure: Revolving a vertical strip about a vertical axis creates a thin cylindrical shell.

Revolution about the x-axis: Method of Shells

If the region is between the x-axis and the positive function $y = f(x)$, and the axis of revolution is the y-axis, then

$$\Delta r = \Delta x, \quad r = x, \quad \text{and} \quad h = f(x).$$

One shell has volume $V = 2\pi x f(x)\Delta x$. The whole solid has volume

$$V = \int_a^b 2\pi x f(x)\,dx$$

A Volume by Shells Demos