Nov 18 Math 2253H sec. 05H Fall 2014

Section 5.3: Method of Cylindrical Shells

Suppose we take a thin strip (with width Δr) that is parallel to the axis of rotation. When we revolve this, we obtain a very thin cylindrical shell with volume

$$V = 2\pi rh\Delta r$$

where r is the average radius (distance between strip and the axis of rotation), and h is the height of the strip.

Revolution about the x-axis: Method of Shells

If the region is between the x-axis and the positive function $y = f(x)$, and the axis of revolution is the y-axis, then

$$\Delta r = \Delta x, \quad r = x, \quad \text{and} \quad h = f(x).$$

One shell has volume $V = 2\pi x f(x) \Delta x$. The whole solid has volume

$$V = \int_a^b 2\pi x f(x)\, dx$$

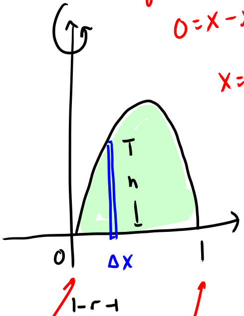
▸ A Volume by Shells Demos

Find the volume of the solid obtained by rotating the first quadrant region bounded between $y = x - x^3$ and the x-axis about the y-axis.

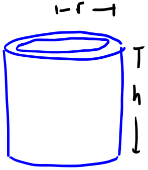
$y = 0$

$0 = x - x^3 = x(1 - x^2)$

$x = 0$ or $x = 1$



x-intercepts



height

$h = y = x - x^3$

radius

$r = x$

Volume of one cylinder

$$V_c = 2\pi x (x - x^3) \Delta x$$

Total volume:

$$V = \int_0^1 2\pi x\,(x - x^3)\,dx$$

$$= 2\pi \int_0^1 (x^2 - x^4)\,dx$$

$$= 2\pi \left[\frac{x^3}{3} - \frac{x^5}{5}\right|_0^1$$

$$= 2\pi \left[\frac{1^3}{3} - \frac{1^5}{5} - \left(\frac{0^3}{3} - \frac{0^5}{5}\right)\right]$$

$$= 2\pi\left(\frac{1}{3}-\frac{1}{5}\right) = 2\pi\left(\frac{5-3}{15}\right)$$

$$= \frac{4\pi}{15}$$

The region bound between $y = \sin(x^2)$ and the x-axis for $0 \leq x \leq \sqrt{\pi}$ is rotated about the y-axis. Find the volume of the resulting solid.

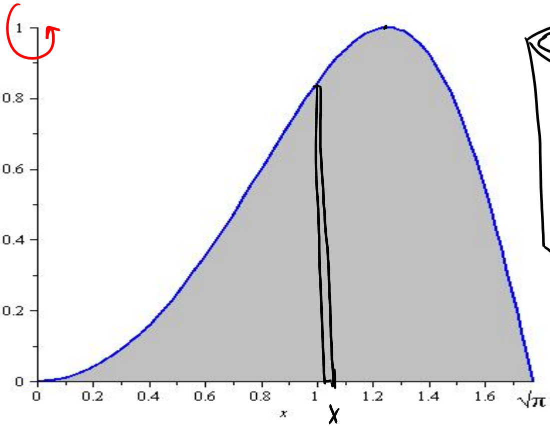
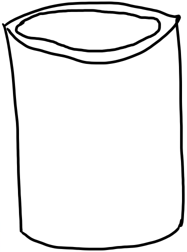


Figure: $y = \sin(x^2)$



height

$h = \sin(x^2)$

radius

$r = x$

Volume of one cylinder

$$V_c = 2\pi x \,\mathrm{Sin}(x^2)\,\Delta x$$

Total Volume

$$V = \int_0^{\sqrt{\pi}} 2\pi x \,\mathrm{Sin}(x^2)\,dx$$

$$= \pi \int_0^{\pi} \mathrm{Sin}(u)\,du$$

Let

$u = x^2$, $du = 2x\,dx$

when $x = 0$

$u = 0^2 = 0$

when $x = \sqrt{\pi}$

$u = (\sqrt{\pi})^2 = \pi$

$$= \pi \left[-\cos u \right|_0^{\pi}$$

$$= \pi \left[-\cos \pi - (-\cos 0) \right]$$

$$= \pi \, (1 + 1)$$

$$= 2\pi$$

Overview Solid Formed by Revolution

Method of Disks or Washers: Take cross sections **perpendicular** to the axis of rotation. If the axis of rotation is the x-axis (or y-axis) then

$$\text{Solid of Revolution:} \quad V = \int_a^b \pi (f(x))^2\, dx \quad \left(\text{or} \quad V = \int_c^d \pi (f(y))^2\, dy\right).$$

This can be adjusted as needed for washers (disk with a disk removed) or for other horizontal or vertical axes of rotation.

Solid Formed by Revolution

Method of Cylindrical Shells: Take cross sections **parallel** to the axis of rotation. For shells of thickness Δr, radius r and height h, the volume is

$$\text{Solid of Revolution:}\quad V = \int_a^b 2\pi rh\,dr.$$

For a vertical axis of rotation, $dr = dx$ and for a horizontal one $dr = dy$. Both r and h must be expressed in terms of the variable of integration.

Solid Formed by Revolution

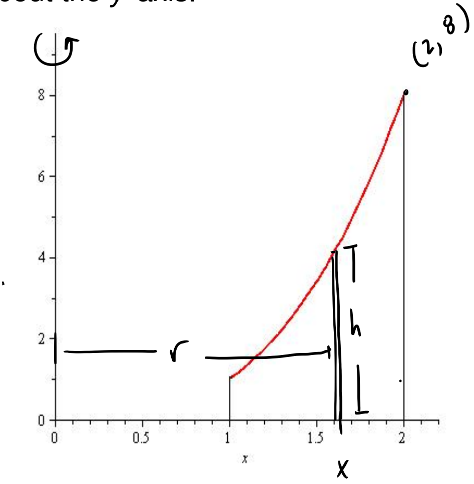
Figure: Performing Arts Center, Kansas City MO

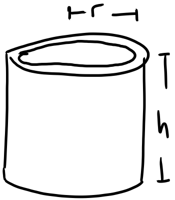
Solid Formed by Revolution

Figure: Performing Arts Center, Kansas City MO

Find the volume of the solid...

The region bound between $y = x^3$, $y = 0$, $x = 1$ and $x = 2$ is rotated about the y-axis.





height $\quad h = y = x^3$

radius $\quad r = x$

Thickness Δx

$$V_c = 2\pi\, x\, x^3\, \Delta x = 2\pi x^4 \Delta x$$

Total Volume:

$$V = \int_1^2 2\pi x^4 \, dx = 2\pi \left. \frac{x^5}{5} \right|_1^2$$

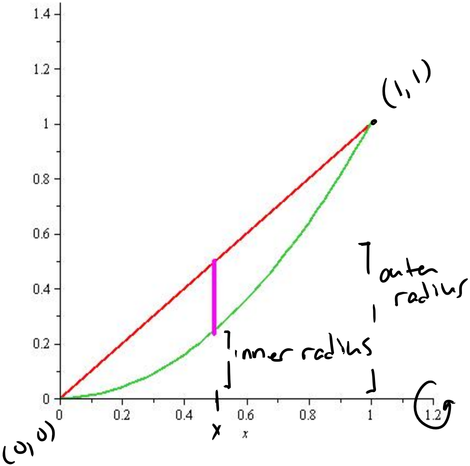
$$= \frac{2\pi}{5}\left(2^5 - 1^5\right) = \frac{2\pi}{5}(32-1) =$$

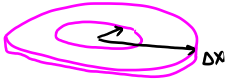
$$= \frac{2\pi}{5}(31)$$

$$= \frac{62\pi}{5}$$

Find the volume of the solid using disks (washers)

The region bound between $y = x$ and $y = x^2$ is rotated about the x-axis.





Inner radius

$R_i = x^2$

Outer radius

$R_o = x$

Figure: Cross section perpedicular to axis for disks/washers.

Volume of one washer

$$V_w = \pi (R_o)^2 \Delta x - \pi (R_i)^2 \Delta x$$

$$= \pi \left((x)^2 - (x^2)^2 \right) \Delta x$$

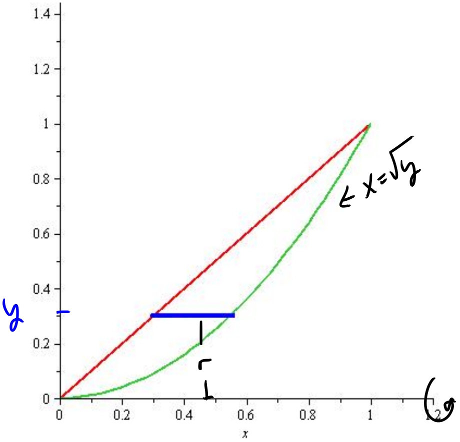
Total Volume

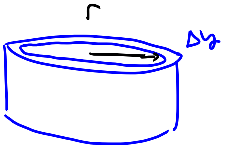
$$V = \int_0^1 \pi (x^2 - x^4)\, dx$$

Exercise: Evaluate this.

Find the volume of the solid using shells.

The region bound between$y = x$ and $y = x^2$ is rotated about the x-axis.





radius $r = y$

height $h = \sqrt{y} - y$

Figure: Cross section parallel to axis for shells.

Volume of one cylinder

$$V_c = 2\pi y(\sqrt{y} - y)\,\Delta y$$

total Volume

$$V = \int_0^1 2\pi y(\sqrt{y} - y)\,dy$$

Exercise: Verify that this value is the same as the one found using washers.