Nov 24 Math 2253H sec. 05H Fall 2014

Section 5.5: Average Value of a Function

For a finite collection of numbers $y_1, y_2, \dots, y_n$, the average (arithmetic) value is the number

$$y_{avg} = \frac{y_1 + y_2 + \cdots + y_n}{n}.$$

We'd like to extend this notation to an infinite collection of numbers $y = f(x)$ for $a \le x \le b$.

If we take a set of sample points $x_1^*, x_2^*, \dots, x_n^*$ for an equally spaced partition of $[a, b]$, we could approximate

$$y_{avg} \approx \frac{f(x_1^*) + f(x_2^*) + \cdots + f(x_n^*)}{n}.$$

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For an equally spaced partition

$$\Delta x = \frac{b-a}{n} \quad \Longrightarrow \quad \frac{1}{n} = \frac{\Delta x}{b-a}.$$

So replacing n we can write

$$y_{avg} \approx \sum_{i=1}^{n} f(x_i^*) \frac{\Delta x}{b-a} = \frac{1}{b-a} \sum_{i=1}^{n} f(x_i^*)\Delta x.$$

We will define the average value of f on the interval $[a, b]$ as the limit of this approximation when $n \to \infty$.

Average Value of a function f on an interval $[a, b]$.

Definition: Provided f is integrable on $[a, b]$, the average value of f on $[a, b]$ is

$$f_{avg} = \frac{1}{b-a}\int_a^b f(x)\,dx.$$

The Mean Value Theorem for Integrals If f is continuous on $[a, b]$, then there exists a number c in (a, b) such that

$$f(c) = f_{avg} = \frac{1}{b-a}\int_a^b f(x)\,dx.$$

In other words, $$\int_a^b f(x)\,dx = f(c)(b-a).$$

MVT for Integrals

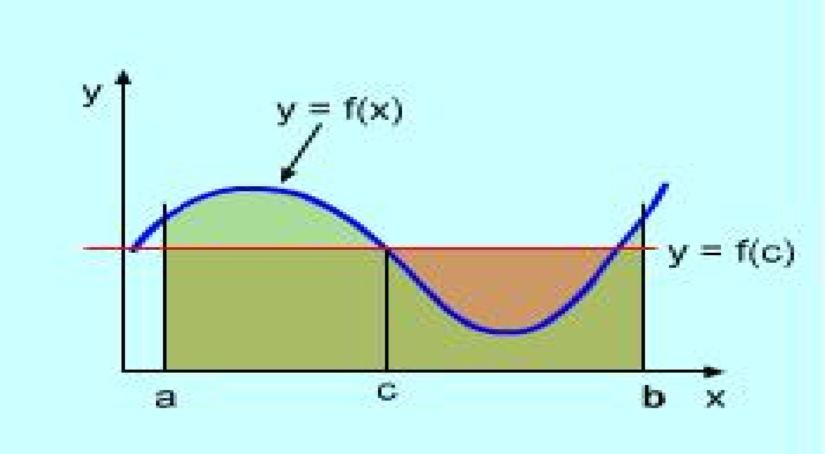


Figure: Mean Value Theorem Illustrated.

Find the average value of $f(x) = x - x^2$ on $[0, 1]$.

$$f_{avg} = \frac{1}{1-0} \int_0^1 f(x)\,dx$$

$$= \int_0^1 (x - x^2)\,dx$$

$$= \frac{x^2}{2} - \frac{x^3}{3} \Big|_0^1 = \left(\frac{1^2}{2} - \frac{1^3}{3}\right) - \left(\frac{0^2}{2} - \frac{0^3}{3}\right)$$

$$= \frac{1}{2} - \frac{1}{3} = \frac{3-2}{6} = \frac{1}{6}$$

Find the average value of $f(x) = \sqrt{x}$ on $[0, 4]$.

Then find the value of c that satifies the conclusion of the MVT for integrals.

$$f_{avg} = \frac{1}{4-0}\int_0^4 f(x)\,dx$$

$$= \frac{1}{4}\int_0^4 x^{1/2}\,dx$$

$$= \frac{1}{4}\left.\frac{x^{3/2}}{3/2}\right|_0^4 = \frac{1}{4}\cdot\frac{2}{3}\left.x^{3/2}\right|_0^4$$

$$= \frac{1}{6}\left[4^{3/2} - 0^{3/2}\right] = \frac{1}{6}[8] = \frac{4}{3}$$

Find a value of c such that $f(c) = f_{avg}$:

$$f(c) = \sqrt{c} = \frac{4}{3} \quad \Rightarrow \quad c = \left(\frac{4}{3}\right)^2 = \frac{16}{9}$$

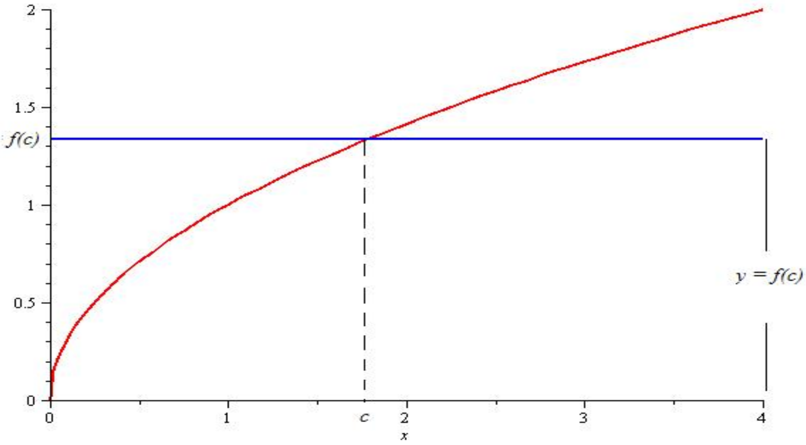


Figure: Mean Value Theorem Illustrated.

Find the average value of $f(x) = \sin x$ on $\left[\frac{\pi}{2}, \frac{3\pi}{2}\right]$.

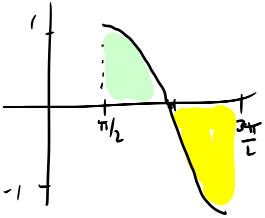
$$f_{avg} = \frac{1}{\frac{3\pi}{2} - \frac{\pi}{2}} \int_{\pi/2}^{3\pi/2} f(x)\, dx$$

$$= \frac{1}{\pi} \int_{\pi/2}^{3\pi/2} \sin x \, dx$$

$$= \frac{1}{\pi} \left[-\cos x \right|_{\pi/2}^{3\pi/2}$$

$$= \frac{1}{\pi} \left[-\cos\left(\frac{3\pi}{2}\right) - \left(-\cos \frac{\pi}{2}\right)\right]$$

$$= \frac{1}{\pi} (0) = 0$$



Find the average value of $g(t) = \tan^2(t)$ on $\left[-\frac{\pi}{4}, \frac{\pi}{4}\right]$.

$$g_{avg} = \frac{1}{\frac{\pi}{4} - \left(\frac{-\pi}{4}\right)} \int_{-\pi/4}^{\pi/4} g(t)\,dt = \frac{2}{\pi} \int_{-\pi/4}^{\pi/4} \tan^2 t \, dt$$

$$= \frac{2}{\pi} \int_{-\pi/4}^{\pi/4} (\sec^2 t - 1)\, dt$$

$$= \frac{2}{\pi} \left[\tan t - t \right|_{-\pi/4}^{\pi/4}$$

$$= \frac{2}{\pi} \left[\tan\frac{\pi}{4} - \frac{\pi}{4} - \left(\tan\frac{-\pi}{4} - \left(\frac{-\pi}{4}\right)\right)\right]$$

$$= \frac{2}{\pi} \left[2 - \frac{\pi}{2} \right] = \frac{4}{\pi} - 1$$

Section 5.4 Work

The physical concept of *work* is the product of the force applied to an object (maginute in the direction of motion) and the distance through which the object is moved.

$$W = Fd$$

For example, if 15 lb of horizontal force is applied to a wagon that is pulled 30 ft along the horizontal, then the work done is

$$W = 15 \times 30\, ft \cdot lb = 450 ft \cdot lb$$

Work with Variable Force

Suppose we wish to move an object along the x-axis from $x = a$ to $x = b$ applying a variable force $F(x)$. We may begin by forming a partition $\{x_0, x_1, \ldots, x_n\}$ of $[a, b]$ and approximating the total work done by assuming that F is constant on each subinterval. We obtain

$$W \approx F(x_1^*)\Delta x + F(x_2^*)\Delta x + \cdots + F(x_n^*)\Delta x$$

We define the work in the limit as $n \to \infty$ to obtain the integral formula

$$W = \lim_{n\to\infty} \sum_{i=1}^{n} F(x_i^*)\Delta x = \int_a^b F(x)\,dx$$

Units

American Standard: Force is given in pounds, and length in feet. The unit for work is the foot-pound (ft lb).

SI (International): Force is given in Newtons (N), and length is in meters. The unit for work is the Newton-meter which is also called a Joule (J).

Example

A particle is moved along the x-axis from the point $x = 2$ to $x = 5$ by applying a force of $F(x) = x^2 - x$. Find the work done.

$$W = \int_a^b F(x)\,dx$$

$$= \int_2^5 (x^2 - x)\,dx = \frac{x^3}{3} - \frac{x^2}{2} \Big|_2^5$$

$$= \frac{5^3}{3} - \frac{5^2}{2} - \left(\frac{2^3}{3} - \frac{2^2}{2}\right)$$

$$= \frac{125}{3} - \frac{25}{2} - \frac{8}{3} + \frac{4}{2} = \frac{117}{3} - \frac{21}{2}$$

$$= 39 - \frac{21}{2} = \frac{78 - 21}{2} = \frac{57}{2}$$

Hooke's Law

Hooke's Law says that

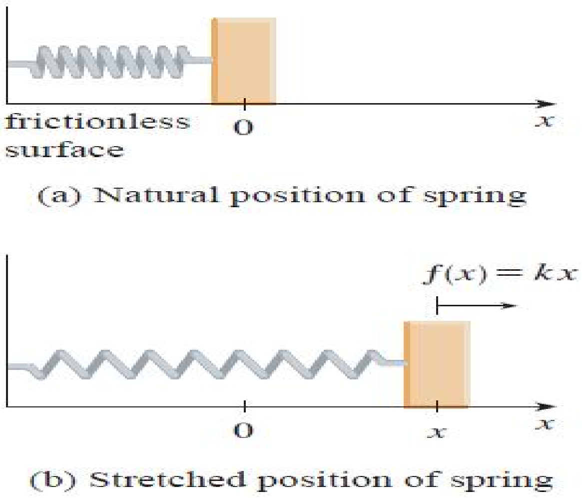
The force exerted on a mass by a spring is proportional to the displacement of the mass from equilibrium.

A spring has a *natural* length in the absence of being stretched or compressed. What Hooke's law says is that to stretch or compress a spring x units from this natural length requires a force

$$F(x) = kx.$$

The value k is called a *spring constant*.

Hooke's Law


frictionless
surface
0
x
(a) Natural position of spring
f(x) = kx
0
x
x
(b) Stretched position of spring

Example

A force of 10 N is required to stretch a spring 25 cm from its equilibrium. Find the work done stretching the spring from its natural length of 5 cm to a length of 50 cm.

Find the force function $F(x) = kx$. We need the value of k. Given

$$10\text{ N} = k(25\text{ cm}) = k(0.25\text{ m})$$

$$\Rightarrow \quad k = \frac{10\text{ N}}{0.25\text{ m}} = 40\ \frac{\text{N}}{\text{m}}$$

So $F(x) = 40x$ N

The natural length 5cm gives $x = 0$ m

The complete stretched length 50cm gives $x = 0.45$ m

Work

$$W = \int_0^{0.45} (40x \text{ N})\, dx \text{ m}$$

$$= \int_0^{0.45} 40x\, dx \text{ J} = 20x^2 \Big|_0^{0.45} \text{ J}$$

$$= 20(0.45)^2 \text{ J} = 4.05 \text{ J}$$

Example

A spring is stretched from its natural length of 1 foot to a length of 3 ft. The work done is 72 ft lb. Determine the spring constant k.

Given $W = 72$ ft·lb, and we know that

$$W = \int_a^b F(x)\,dx = \int_0^2 kx\,dx \quad \text{ft}\cdot\text{lb}$$

$$72\ \text{ft}\cdot\text{lb} = k\,\frac{x^2}{2}(\text{ft})^2\Big|_0^2 = k\left(\frac{4}{2} - \frac{0}{2}\right)\text{ft}^2$$

$$72\ \text{ft}\cdot\text{lb} = 2k\ \text{ft}^2 \implies k = \frac{72\ \text{ft}\cdot\text{lb}}{2\ \text{ft}^2} = 36\ \frac{\text{lb}}{\text{ft}}$$