Nov 25 Math 2253H sec. 05H Fall 2014

Section 5.4 Work

The work done by a variable force $F(x)$ to move an object along the straight line from $x = a$ to $x = b$ is

$$W = \int_a^b F(x)\,dx$$

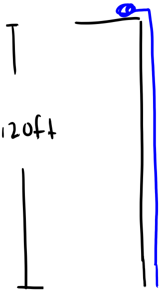
Units: The Imperial (US) unit of work is the foot-pound (*ft lb*), and the SI unit of work is the Newton-meter (*N m*) also called the Joule (*J*).

Anoher Work Example

A chain weighing 2 lb/ft is used to haul a 200 lb girder to the top of a building 120 ft high. Find the work done lifting the girder to the top of the building.

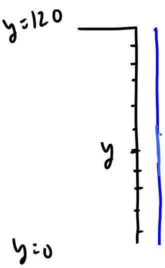
Total work is that done on the girder W_g

plus that done on the chain W_c.



work $W = W_g + W_c$

$$W_g = 120\,\text{ft} \cdot 200\,\text{lb} = 24000\ \text{ft}\cdot\text{lb}$$



Cut the axis into pieces of thickness Δy

Take a length Δy of chain @ some y between 0 and 120.

Call the force and distance for our piece F_p and D_p $\Rightarrow W_p = F_p \cdot D_p$

$$F_p = \Delta y \text{ ft}\left(2\,\frac{\text{lb}}{\text{ft}}\right) = 2\Delta y \text{ lb}, \quad D_p = (120-y)\text{ ft}$$

$$W_p = (2\Delta y \text{ lb})\cdot(120-y)\text{ ft} = 2(120-y)\Delta y \text{ ft lb}$$

Sum these from $y=0$ to $y=120$ in the limit.

$$W = \int_0^{120} 2(120-y)\,dy \quad ft \cdot lb$$

$$= 2\int_0^{120} (120-y)\,dy \quad ft \cdot lb$$

$$= 2\left[120y - \frac{y^2}{2}\right|_0^{120} \quad ft \cdot lb$$

$$= 2\left[120 \cdot 120 - \frac{120^2}{2} - \left(120 \cdot 0 - \frac{0^2}{2}\right)\right] \quad ft \cdot lb$$

$$= 2(120^2)\left(1 - \frac{1}{2}\right) \quad ft \cdot lb$$

$$= 2(120^2)\left(\frac{1}{2}\right) \quad ft \cdot lb$$

$= 120^2 \text{ ft lb} = 14{,}400 \text{ ft lb}$

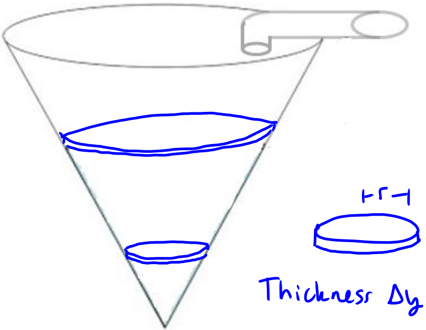
The total work done lifting girder with the chain is

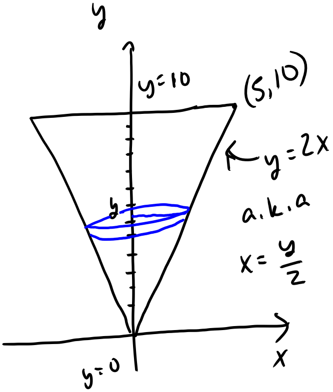
$$W = 24{,}000 \text{ ft}\cdot\text{lb} + 14{,}400 \text{ ft}\cdot\text{lb}$$

$$= 38{,}400 \text{ ft}\cdot\text{lb}$$

A Final Work Example

A tank of water in the shape of an inverted right circular cone is to be drained by pumping fluid through an opening at the top. The height and base radius of the cone are $h = 10$ ft and $r = 5$ ft, respectively. If water weighs 62 lb/ft^3, determine the work done emptying the tank.





$r = x = \frac{y}{2}$

Again let F_p and D_p be the force and distance for one such piece of fluid.

$$F_p = \text{Volume} \times \text{density}$$
$$= \frac{\pi}{4} y^2 \Delta y \text{ ft}^3 \left(62 \frac{\text{lb}}{\text{ft}^3}\right)$$
$$= \frac{31\pi}{2} y^2 \Delta y \text{ lb}$$

$$\text{Volume} = \pi r^2 \Delta y \text{ ft}^3$$
$$= \pi \left(\frac{y}{2}\right)^2 \Delta y \text{ ft}^3$$
$$= \frac{\pi}{4} y^2 \Delta y \text{ ft}^3$$

$$D_p = (10 - y) \text{ ft}, \quad W_p = \frac{31\pi}{2} y^2 \Delta y \text{ lb } (10-y) \text{ ft}$$

Sum these in the limit to get

$$W = \int_0^{10} \frac{31\pi}{2} y^2 (10-y)\, dy \qquad ft \cdot lb$$

$$= \frac{31\pi}{2} \int_0^{10} (10y^2 - y^3)\, dy \qquad ft \cdot lb$$

$$= \frac{31\pi}{2} \left[10 \frac{y^3}{3} - \frac{y^4}{4} \right|_0^{10} \qquad ft \cdot lb$$

$$= \frac{31\pi}{2} \left[10 \cdot \frac{10^3}{3} - \frac{10^4}{4} - \left(10 \cdot \frac{0^3}{3} - \frac{0^4}{4} \right) \right] \quad ft \cdot lb$$

$$= \frac{31\pi}{2}(10^4)\left(\frac{1}{3} - \frac{1}{4}\right) \quad ft \cdot lb$$

$$= \frac{31\pi}{2}(10^4)\left(\frac{4-3}{12}\right) \quad ft \cdot lb$$

$$= \frac{31\pi(10^4)}{24} \quad ft \cdot lb$$

$$\approx 40,580 \quad ft \cdot lb$$

F  *N*