# Nov 3 Math 2253H sec. 05H Fall 2014

## Section 4.5: The Substitution Rule

**Theorem:** Suppose $u = g(x)$ is a differentiable function, and $f$ is continuous on the range of $g$. Then

$$\int f(g(x))\, g'(x)\, dx = \int f(u)\, du.$$

This is often refered to as ***u*-substitution.**
This is the Chain Rule in reverse!

# Evaluate each Indefinite integral using Substitution as Needed

(b) $\displaystyle\int t\sec^2(t^2)\,dt$

Let $u = t^2$

$du = 2t\,dt$

Note $du = 2t\,dt \Rightarrow$

$\frac{1}{2}du = t\,dt$

$$= \int \frac{1}{2}\sec^2(t^2)\,2t\,dt$$

$$= \frac{1}{2}\int \sec^2(u)\,du = \frac{1}{2}\tan(u) + C$$

$$= \frac{1}{2}\tan(t^2) + C$$

(c) $\int \frac{\cos\sqrt{x}}{\sqrt{x}}\,dx = \int \cos\sqrt{x} \cdot \frac{1}{\sqrt{x}}\,dx$

Let

$u = \sqrt{x}$

$du = \frac{1}{2\sqrt{x}}\,dx$

$\Rightarrow 2du = \frac{1}{\sqrt{x}}\,dx$

$= \int \cos u \; 2\,du$

$= 2\int \cos u \, du$

$= 2\sin u + C$

$= 2\sin\sqrt{x} + C$

(d) $\displaystyle\int \frac{\csc^2 x}{\cot^2 x}\,dx = \int (\cot x)^{-2} \csc^2 x \, dx$

Let

$u = \cot x$

$du = -\csc^2 x \, dx$

$\Rightarrow -du = \csc^2 x \, dx$

$= -\displaystyle\int u^{-2} \, du$

$= -\dfrac{u^{-1}}{-1} + C$

$= \dfrac{1}{u} + C = \dfrac{1}{\cot x} + C$

$= \tan x + C$

# A Subtle use of Substitution

Evaluate

$$\int x\sqrt{x+1}\,dx \quad \text{by taking} \quad u = x+1$$

If we had $(x+1)\sqrt{x} = x^{3/2} + x^{1/2}$.

Let $u = x+1$, $du = dx$

subtracting 1 gives $x = u-1$

$$\int x\sqrt{x+1}\,dx = \int (u-1)\sqrt{u}\,du$$

$$= \int (u-1)u^{1/2}\,du$$

$$= \int (u^{3/2} - u^{1/2})\,du$$

$$= \frac{u^{5/2}}{5/2} - \frac{u^{3/2}}{3/2} + C$$

$$= \frac{2}{5}u^{5/2} - \frac{2}{3}u^{3/2} + C$$

$$= \frac{2}{5}(x+1)^{5/2} - \frac{2}{3}(x+1)^{3/2} + C$$

# A Subtle use of Substitution

Evaluate

$$\int x\sqrt{x+1}\,dx \quad \text{by taking} \quad u = \sqrt{x+1}$$

Note that $u^2 = x+1$. $x = u^2 - 1$ so $dx = 2u\,du$

$$\int x\sqrt{x+1}\,dx = \int (u^2-1)u \cdot 2u\,du$$

$$= 2\int (u^2-1)u^2\,du$$

$$= 2\int (u^4 - u^2)\,du$$

$$= 2\left(\frac{u^5}{5} - \frac{u^3}{3}\right) + C$$

$$= \frac{2}{5}u^5 - \frac{2}{3}u^3 + C$$

$$= \frac{2}{3}\left(\sqrt{x+1}\right)^5 - \frac{2}{3}\left(\sqrt{x+1}\right)^3 + C$$

$$= \frac{2}{3}(x+1)^{5/2} - \frac{2}{3}(x+1)^{3/2} + C$$

# Theorem (Substitution for Definite Integrals)

Suppose $g'$ is continuous on $[a, b]$ and $f$ is continuous on the range of $u = g(x)$. Then

$$\int_a^b f(g(x))\, g'(x)\, dx = \int_{g(a)}^{g(b)} f(u)\, du$$

## Evaluate each Definite Integral

(a) $\int_0^1 x\sqrt{1-x^2}\,dx$

$$= \int_0^1 \sqrt{1-x^2}\; x\,dx$$

$$= \int_1^0 u\,(-u)\,du$$

$$= -\int_1^0 u^2\,du$$

Let $u = \sqrt{1-x^2}$

$= (1-x^2)^{1/2}$

$du = \frac{1}{2}(1-x^2)^{-1/2}(-2x)dx$

$= \dfrac{-x}{\sqrt{1-x^2}}\,dx$

$du = \frac{-x}{u}\,dx$

$-u\,du = x\,dx$

when $x=0$, $u=\sqrt{1-0^2}=1$

when $x=1$, $u=\sqrt{1-1^2}=0$

$$= -\frac{u^3}{3}\Big|_1^0 = \frac{-0^3}{3} - \left(\frac{-1^3}{3}\right)$$

$$= 0 - \left(\frac{-1}{3}\right) = \frac{1}{3}$$

$$u = 1 - x^2, \quad du = -2x\,dx \implies \frac{-1}{2}du = x\,dx$$

$$\int_0^1 x\sqrt{1-x^2}\,dx = \frac{-1}{2}\int_1^0 \sqrt{u}\,du$$

# A Geometric Interpretation

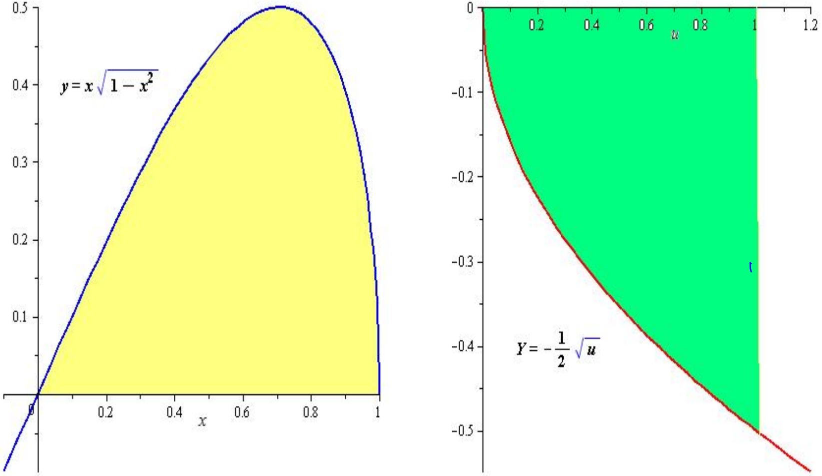


Figure: The yellow region's area is $\frac{1}{3}$. The green region also has area $\frac{1}{3}$.

(b) $\int_0^{\pi/4} \cos\left(2t - \frac{\pi}{4}\right)\, dt$

Let $u = 2t - \frac{\pi}{4}$

$du = 2\, dt$

$\frac{1}{2}\, du = dt$

When $t = 0$

$u = 2 \cdot 0 - \frac{\pi}{4} = \frac{-\pi}{4}$

When $t = \frac{\pi}{4}$

$u = 2 \cdot \frac{\pi}{4} - \frac{\pi}{4} = \frac{\pi}{4}$

$$= \int_{-\frac{\pi}{4}}^{\pi/4} \frac{1}{2} \cos u \, du$$

$$= \frac{1}{2} \sin u \Big|_{-\pi/4}^{\pi/4}$$

$$= \frac{1}{2}\sin\frac{\pi}{4} - \frac{1}{2}\sin\left(\frac{-\pi}{4}\right)$$

$$= \frac{1}{2}\frac{1}{\sqrt{2}} - \frac{1}{2}\left(\frac{-1}{\sqrt{2}}\right) = \frac{1}{2\sqrt{2}} + \frac{1}{2\sqrt{2}} = \frac{2}{2\sqrt{2}} = \frac{1}{\sqrt{2}}$$

$$\int \cos(2t - \pi/4)\,dt = \frac{1}{2}\int \cos u\,du = \frac{1}{2}\sin u + C$$

$$= \frac{1}{2}\sin\left(2t - \frac{\pi}{4}\right) + C$$

$$\int_0^{\pi/4} \cos\left(2t - \frac{\pi}{4}\right)dt = \frac{1}{2}\sin\left(2t - \frac{\pi}{4}\right)\Big|_0^{\pi/4}$$

$$= \frac{1}{2}\sin\left(2\cdot\frac{\pi}{4} - \frac{\pi}{4}\right) - \frac{1}{2}\sin\left(2\cdot 0 - \frac{\pi}{4}\right) = \frac{1}{2}\sin\frac{\pi}{4} - \frac{1}{2}\sin\left(\frac{-\pi}{4}\right)$$

$$= \frac{1}{\sqrt{2}}$$