Nov 4 Math 2253H sec. 05H Fall 2014

Section 4.5: The Substitution Rule

Theorem (u-Substitution): Suppose $u = g(x)$ is a differentiable function, and f is continuous on the range of g. Then

$$\int f(g(x))\, g'(x)\, dx = \int f(u)\, du.$$

Theorem (Substitution for Definite Integrals): Suppose g' is continuous on $[a, b]$ and f is continuous on the range of $u = g(x)$. Then

$$\int_a^b f(g(x))\, g'(x)\, dx = \int_{g(a)}^{g(b)} f(u)\, du$$

Evaluate each Definite Integral

(c) $\int_{-1}^{3} y\sqrt{y+1}\,dy$

Let $u = y+1$

$du = dy$

$y = u-1$

when $y=-1$,

$u = -1+1 = 0$

when $y = 3$

$u = 3+1 = 4$

$$= \int_0^4 (u-1)\sqrt{u}\,du$$

$$= \int_0^4 \left(u^{3/2} - u^{1/2}\right) du$$

$$= \left. \frac{u^{5/2}}{5/2} - \frac{u^{3/2}}{3/2} \right|_0^4$$

$$= \frac{2}{5} u^{5/2} - \frac{2}{3} u^{3/2} \Big|_0^4$$

$$= \frac{2}{5}(4)^{5/2} - \frac{2}{3}(4)^{3/2} - \left(\frac{2}{5}(0)^{5/2} - \frac{2}{3}(0)^{3/2} \right)$$

$$= \frac{2}{5}(32) - \frac{2}{3}(8)$$

$$= 16\left(\frac{4}{5} - \frac{1}{3}\right) = 16\left(\frac{12-5}{15}\right) = \frac{16 \cdot 7}{15} = \frac{112}{15}$$

(d) $\displaystyle\int_0^{\pi/4} \sec^2 t \tan t \, dt$

Let $u = \tan t$

$du = \sec^2 t \, dt$

If $t = 0$, $u = \tan(0) = 0$

$t = \frac{\pi}{4}$, $u = \tan\left(\frac{\pi}{4}\right) = 1$

$$= \int_0^1 u \, du$$

$$= \left. \frac{u^2}{2} \right|_0^1$$

$$= \frac{1^2}{2} - \frac{0^2}{2} = \frac{1}{2}$$

$$\int_0^{\pi/4} \sec^2 t \tan t \, dt = \int_0^{\pi/4} \sec t \sec t \tan t \, dt$$

Let $u = \sec t$

$du = \sec t \tan t \, dt$

If $t = 0$, $u = \sec(0) = 1$

$t = \frac{\pi}{4}$, $u = \sec\left(\frac{\pi}{4}\right) = \sqrt{2}$

$$= \int_1^{\sqrt{2}} u \, du$$

$$= \left.\frac{u^2}{2}\right|_1^{\sqrt{2}}$$

$$= \frac{(\sqrt{2})^2}{2} - \frac{1^2}{2} = \frac{2}{2} - \frac{1}{2} = \frac{1}{2}$$

Symmetry and Integrals from $-a$ to a

Recall: A function f is even if $f(-x) = f(x)$.
A function f is odd if $f(-x) = -f(x)$.

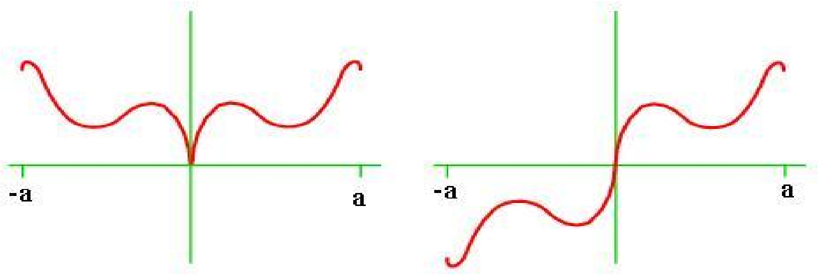

Figure: Symmetric Functions: Left is even, right is odd.

Theorem:

If f is an even, integrable function, then

$$\int_{-a}^{a} f(x)\,dx = 2\int_{0}^{a} f(x)\,dx.$$

If f is an odd, integrable function, then

$$\int_{-a}^{a} f(x)\,dx = 0.$$

Symmetry and Integrals from $-a$ to a

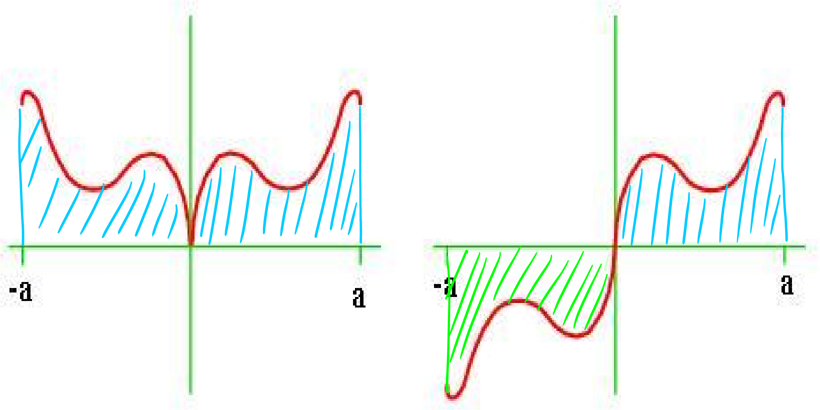

Figure: $\int_{-a}^{a} f(x)\,dx = 2\int_0^a f(x)\,dx$ if f is even. $\int_{-a}^{a} f(x)\,dx = 0$ if f is odd.

Evaluate

$$\int_{-2.3}^{2.3} x^7 \cos(4x)\, dx$$

$f(x) = x^7 \cos(4x)$, $f(-x) = (-x)^7 \cos(-4x)$

$= (-1)^7 x^7 \cos(4x)$

$= -x^7 \cos(4x) = -f(x)$

$\cos(x)$ is even

f is odd

$$\int_{-2.3}^{2.3} x^7 \cos(4x)\, dx = 0$$

Evaluate

$$\int_{-3}^{3} (x^2+1)\,dx$$

$f(x) = x^2+1$, $f(-x) = (-x)^2+1$

$= x^2+1 = f(x)$

f is even

Using symmetry

$$\int_{-3}^{3} (x^2+1)\,dx = 2\int_{0}^{3} (x^2+1)\,dx$$

$$= 2\left(\frac{x^3}{3}+x\right)\Big|_0^3 = 2\left(\frac{3^3}{3}+3\right) - 2\left(\frac{0^3}{3}+0\right)$$

$$= 2(9+3) = 2\cdot 12 = 24$$

$$\int_{-3}^{3} (x^2+1)\,dx = \left.\frac{x^3}{3} + x \right|_{-3}^{3}$$

$$= \frac{3^3}{3} + 3 - \left(\frac{(-3)^3}{3} + (-3) \right)$$

$$= 9 + 3 - (-9 - 3)$$

$$= 12 - (-12) = 24$$

Your Turn!

Evaluate $\displaystyle\int_0^1 x^2(x^3+3)^4\,dx$ $= \dfrac{781}{15}$

Let $u = x^3 + 3$ so $\frac{1}{3}du = x^2 dx$

and the integral becomes

$$\frac{1}{3}\int_3^4 u^4\,du = \frac{1}{3}\frac{u^5}{5}\Big|_3^4 = \frac{781}{15}$$

Your Turn!

Evaluate $\int \frac{\sin\left(\frac{1}{x}\right)}{x^2}\,dx$ $=$ $\cos\left(\frac{1}{x}\right) + C$

Let $u = \frac{1}{x}$ so $du = \frac{-1}{x^2}\,dx$ to get

$$\int -\sin(u)\,du = \cos(u) + C = \cos\left(\frac{1}{x}\right) + C$$