Nov 6 Math 2253H sec. 05H Fall 2014

Section 5.1: Area Between Curves

Consider a pair of continuous curves $y = f(x)$ and $y = g(x)$ for $a \leq x \leq b$.

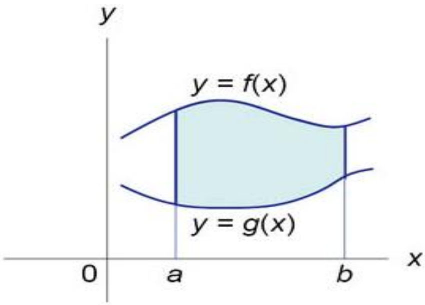


Figure: The curves enclose a region. We can ask what its area is.

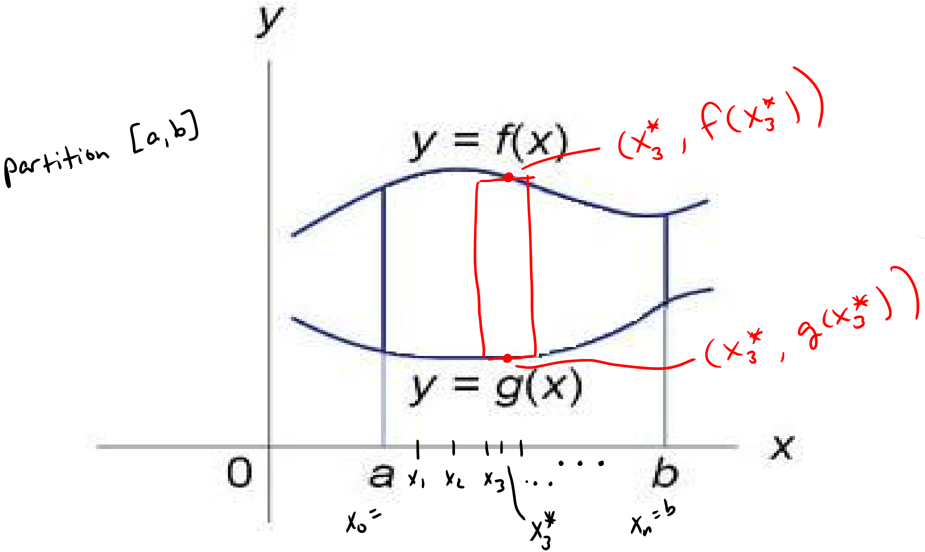


Figure: We can "build" the area from approximating rectangles.

height of rectangle $h = f(x_3^*) - g(x_3^*)$

width of rectangle $w = \Delta x$

area of one rectangle $hw = \left(f(x_3^*) - g(x_3^*)\right)\Delta x$

$$A \approx \sum_{i=1}^{n} \left(f(x_i^*) - g(x_i^*)\right)\Delta x$$

Exact area

$$A = \lim_{n\to\infty} \sum_{i=1}^{n} \left(f(x_i^*) - g(x_i^*)\right)\Delta x$$

$$= \int_a^b \left(f(x) - g(x)\right)dx$$

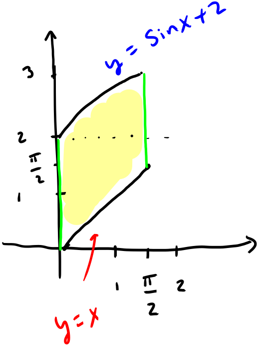
Area Between Curves:

Suppose f and g are continuous on $[a, b]$ and $f(x) \geq g(x)$. The area A bounded between the curves $y = f(x)$, $y = g(x)$ and the lines $x = a$ and $x = b$ is

$$A = \int_a^b (f(x) - g(x))\, dx.$$

Example

Determine the area bounded between the curves $y = \sin(x) + 2$ and $y = x$ on $[0, \pi/2]$.



$$A = \int_0^{\pi/2} \left(\sin x + 2 - x \right) dx$$

$$= -\cos x + 2x - \frac{x^2}{2} \Big|_0^{\pi/2}$$

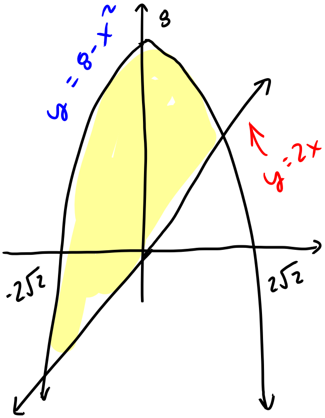
$$= \left(-\cos\frac{\pi}{2} + 2\left(\frac{\pi}{2}\right) - \frac{(\pi/2)^2}{2} \right) - \left(-\cos 0 + 0 - 0 \right)$$

$$= \pi - \frac{\pi^2}{8} + 1$$

$$= \frac{8\pi - \pi^2 + 8}{8}$$

Example

Determine the area bounded between the curves $y = 8 - x^2$ and $y = 2x$.



Find points of intersection:

$$8 - x^2 = 2x \Rightarrow$$

$$0 = x^2 + 2x - 8$$

$$0 = (x - 2)(x + 4)$$

$$\Rightarrow x = 2 \quad \text{or} \quad x = -4$$

area

$$A = \int_{-4}^{2} (8 - x^2 - 2x)\, dx$$

$$= 8x - \frac{x^3}{3} - x^2 \Big|_{-4}^{2}$$

$$= \left(16 - \frac{8}{3} - 4\right) - \left(-32 + \frac{64}{3} - 16\right)$$

$$= 12 - \frac{8}{3} + 48 - \frac{64}{3} = 60 - \frac{72}{3} = 60 - 24$$

$$= 36$$

The condition $f(x) \geq g(x)$

Consider the curves $y = x^3$ and $y = x$. These *bound* a region, but the curves switch places.

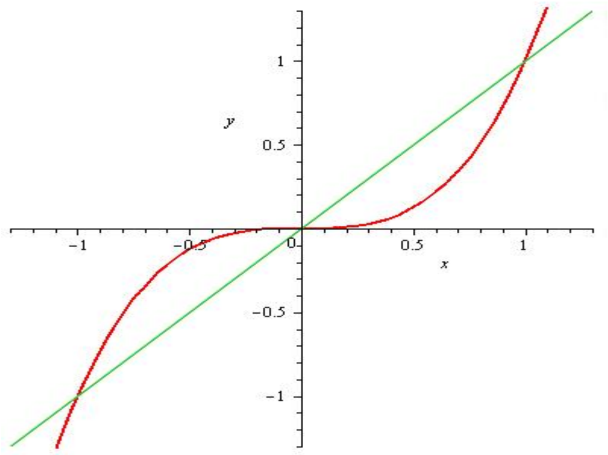


Figure: Curves $y = x^3$ and $y = x$.

Area Between Curves:

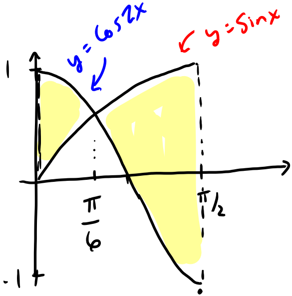
Suppose f and g are continuous on $[a,b]$. The area A bounded between the curves $y=f(x)$, $y=g(x)$ and the lines $x=a$ and $x=b$ is

$$A=\int_a^b |f(x)-g(x)|\,dx.$$

Example

Find the area bounded between the curves $y = \cos 2x$, $y = \sin x$, $x = 0$ and $x = \frac{\pi}{2}$.

$$A = \int_0^{\pi/2} |\cos 2x - \sin x|\, dx$$



Find the intersection:

$$\cos 2x = \sin x$$

$$1 - 2\sin^2 x = \sin x$$

$$0 = 2\sin^2 x + \sin x - 1$$

$$0 = (2\sin x - 1)(\sin x + 1)$$

$$\Rightarrow \quad \sin x = \frac{1}{2} \quad \text{or} \quad \sin x = -1$$

no solution on $[0, \frac{\pi}{2}]$

$$\sin x = \frac{1}{2} \Rightarrow x = \frac{\pi}{6}$$

for $0 \le x \le \frac{\pi}{2}$

$$A = \int_0^{\pi/2} |\cos 2x - \sin x| \, dx$$

$$= \int_0^{\pi/6} (\cos 2x - \sin x) \, dx + \int_{\pi/6}^{\pi/2} (\sin x - \cos 2x) \, dx$$

$$= \frac{1}{2} \sin 2x + \cos x \Big|_0^{\pi/6} + \Big[-\cos x - \frac{1}{2} \sin 2x \Big|_{\pi/6}^{\pi/2}$$

$$= \frac{1}{2}\sin\frac{\pi}{3} + \cos\frac{\pi}{6} - \left(\frac{1}{2}\sin 0 + \cos 0\right) + \left[-\cos\frac{\pi}{2} - \frac{1}{2}\sin\pi - \left(-\cos\frac{\pi}{6} - \frac{1}{2}\sin\frac{\pi}{3}\right)\right]$$

$$= \frac{\sqrt{3}}{4} + \frac{\sqrt{3}}{2} - 1 + \left[\frac{\sqrt{3}}{2} + \frac{\sqrt{3}}{4}\right]$$

$$= \sqrt{3} + \frac{\sqrt{3}}{2} - 1 \quad = \frac{3\sqrt{3}}{2} - 1$$

Let $u = 2x$ $\quad du = 2dx$

$\frac{1}{2}du = dx$

$$\int \cos 2x \, dx = \int \frac{1}{2}\cos u \, du$$

$$= \frac{1}{2}\sin u + C = \frac{1}{2}\sin 2x + C$$

Horizontal Orientation

A region may be better described by curves $x = f(y)$ and $x = g(y)$, $c \leq y \leq d$

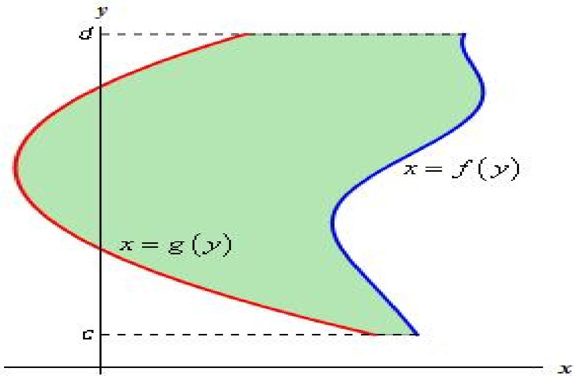


Figure: A horizontally oriented region.

$$A = \int_c^d |f(y) - g(y)|\, dy$$