Oct 13 Math 2253H sec. 05H Fall 2014

Homework Problem (sec. 3.3): Show that the inflection points of the curve $y = x \sin x$ lie on the curve defined by

$$y^2(x^2+4) = 4x^2.$$

We found that $y'' = 2\cos x - x \sin x$. So that an inflection point would require

$$y'' = 0 \quad \Longrightarrow \quad 2\cos x = x \sin x \quad \textbf{i.e.} \quad x = 2\cot x.$$

Since $y = x \sin x$, the condition $y'' = 0$ can also be interpreted as requiring

$$2\cos x = y.$$

$$y^2(x^2+4) = 4x^2$$

$x = 2\cot x \implies x^2 = 4\cot^2 x$ so $x^2+4 = 4\cot^2 x + 4$

$$= 4(\cot^2 x + 1)$$

$$= 4\csc^2 x$$

Hence

$$y^2(x^2+4) = (2\cos x)^2\, 4\csc^2 x$$

$$= 4\cos^2 x \; 4\csc^2 x$$

$$= 16\,\frac{\cos^2 x}{\sin^2 x}$$

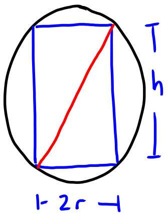
$$= 16\cot^2 x$$

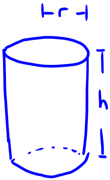
$$= 4(4\cot^2 x) = 4x^2$$ as required.

Section 3.7: Applied Optimization

Find the volume of the largest right circular cylinder that can be inscribed in a sphere of radius 10.

Slice the sphere along a great circle and look at the cross section



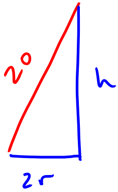


Let r and h be the radius and height of the cylinder.

Volume

$$V = \pi r^2 h$$

objective



$$(2r)^2 + h^2 = 20^2$$ constraint

$$4r^2 + h^2 = 20^2$$

$$4r^2 = 20^2 - h^2 \Rightarrow r^2 = \frac{1}{4}\left(20^2 - h^2\right)$$

So $$V = \pi\left(\frac{1}{4}\left(20^2 - h^2\right)\right)h = \frac{\pi}{4}\left(20^2 h - h^3\right), \quad 0 < h < 20$$

Find crit #:

$$V'(h) = \frac{\pi}{4}\left(20^2 - 3h^2\right)$$

$V'(h)$ is never undefined

$$V'(h) = 0 \Rightarrow \frac{\pi}{4}\left(20^2 - 3h^2\right) = 0$$

$$\Rightarrow \quad 20^2 - 3h^2 = 0 \quad \Rightarrow \quad h^2 = \frac{20^2}{3}$$

$$\Rightarrow \quad h = \frac{20}{\sqrt{3}} \quad \text{or} \quad h = \frac{-20}{\sqrt{3}}$$ ignore since $h > 0$

Use 2nd der. test to see if we have a maximizer.

$$V''(h) = \frac{\pi}{4}(-6h) = \frac{-3\pi}{2} h$$

$$V''\left(\frac{20}{\sqrt{3}}\right) = \frac{-3\pi}{2}\left(\frac{20}{\sqrt{3}}\right) < 0$$

V is concave down @ $\frac{20}{\sqrt{3}}$. we have a max.

$$V\left(\frac{20}{\sqrt{3}}\right) = \frac{\pi}{4}\left(20^2\left(\frac{20}{\sqrt{3}}\right) - \left(\frac{20}{\sqrt{3}}\right)^3\right)$$

$$= \frac{\pi}{4}\left(\frac{20^3}{\sqrt{3}} - \frac{20^3}{3\sqrt{3}}\right)$$

$$= \frac{20^3\pi}{4\sqrt{3}}\left(1 - \frac{1}{3}\right)$$

$$= \frac{20^3\pi}{4\sqrt{3}}\left(\frac{2}{3}\right) = \frac{20^3\pi}{6\sqrt{3}} \approx 2418.4$$

The maximum volume is about 2418.4 cubic units.

Your Turn!

The sum of twice a number and a second number is 32. Find the maximum value of the product of these two numbers.

Calling the numbers k and v

$$2k + v = 32$$

The product $S = kv$.

From the top equation

$$v = 32 - 2k$$

So $\quad S = k(32-2k) = 32k - 2k^2$

$S'(k) = 32 - 4k$, S' is always defined

$S'(k) = 0 \Rightarrow 32 - 4k = 0$

$\Rightarrow k = 8$

$S''(k) = -4$ so $S''(8) = -4 < 0$

we have a max

$k = 8 \Rightarrow \quad v = 32 - 2\cdot 8 = 16$

The maximum product is

$$\not{S} = 8(16) = 128$$

Section 3.9 Antiderivatives

a key word

Definition: A function F is called an antiderivative of f on an interval I if

$$F'(x) = f(x) \quad \text{for all } x \text{ in } I.$$

For example, $F(x) = x^2$ is an antiderivative of $f(x) = 2x$ on $(-\infty, \infty)$. Similarly, $G(x) = \tan x + 7$ is an antiderivative of $g(x) = \sec^2 x$ on $(-\pi/2, \pi/2)$.

Theorem: If F is any antiderivative of f on an interval I, then the *most general* antiderivative of f on I is

$$F(x) + C \quad \text{where } C \text{ is an arbitrary constant.}$$

Find the most general antiderivative of f.

(a) $f(x) = \cos x \quad I = (-\infty, \infty)$

$$F(x) = \sin x + C$$

(b) $f(x) = \sin x \quad I = (-\infty, \infty)$

$$F(x) = -\cos x + C$$

Find the most general antiderivative of f.

(c) $f(x) = \csc x \cot x \quad I = (-\pi, 0)$

$F(x) = -\csc x + C$

Power rule

$\frac{d}{dx} x^n = n x^{n-1}$

(d) $f(x) = \dfrac{1}{x^3} \quad I = (0, \infty)$

$f(x) = x^{-3}$

we'd need

$n - 1 = -3$

$n = -2$

A candidate is

$F(x) = A x^{-2}$ for some constant A

$F'(x) = -2A x^{-3} = x^{-3}$ if $-2A = 1$

$F(x) = \frac{-1}{2} x^{-2} + C$