Oct 14 Math 2253H sec. 05H Fall 2014

Section 3.9 Antiderivatives

Definition: A function F is called an antiderivative of f on an interval I if

$$F'(x) = f(x) \quad \text{for all } x \text{ in } I.$$

Theorem: If F is any antiderivative of f on an interval I, then the *most general* antiderivative of f on I is

$$F(x) + C \quad \text{where } C \text{ is an arbitrary constant.}$$

Find the most general antiderivative of

$$f(x) = x^n, \quad \text{where } n = 1, 2, 3, \dots$$

power rule for derivatives

$$\frac{d}{dx} Ax^k = kA x^{k-1}$$

A, k are constant

we want $kA x^{k-1} = f(x) = x^n$

This requires $k - 1 = n$ and

$kA = 1$

So $k = n+1$ and $(n+1)A = 1 \Rightarrow A = \frac{1}{n+1}$

This gives $F(x) = \frac{1}{n+1} x^{n+1} + C$

Some general results[1]:

Function	Particular Antiderivative	Function	Particular Antiderivative
$cf(x)$	$cF(x)$	$\cos x$	$\sin x$
$f(x)+g(x)$	$F(x)+G(x)$	$\sin x$	$-\cos x$
$x^n,\ n\neq -1$	$\frac{x^{n+1}}{n+1}$	$\sec^2 x$	$\tan x$

power rule for antiderivatives

Find the most general antiderivative of $h(x) = x\sqrt{x}$ on $(0,\infty)$.

$h(x) = x^{3/2}$

$H(x) = \dfrac{x^{3/2+1}}{\frac{3}{2}+1} + C = \dfrac{2}{5}x^{5/2} + C$

[1] We'll use the term particular antiderivative to refer to any antiderivative that has no arbitrary constant in it.

Example

Determine the function $H(x)$ that satisfies the following conditions

$$H'(x) = x\sqrt{x}, \quad \text{for all } x > 0, \text{ and } H(1) = 0.$$

From the last example, all functions such that $H'(x) = x\sqrt{x}$ have the form

$$H(x) = \frac{2}{5} x^{5/2} + C$$

impose $H(1) = 0$

set $x = 1$ and $H = 0$

$$0 = H(1) = \frac{2}{5}(1)^{5/2} + C$$

$$0 = \frac{2}{5} + C \Rightarrow C = \frac{-2}{5}$$

$$\boxed{\text{So} \quad H(x) = \frac{2}{5} x^{5/2} - \frac{2}{5}}$$

Example

A particle moves along the x-axis so that its acceleration at time t is given by

$$a(t) = 12t - 2 \quad \text{m/sec}^2.$$

At time $t = 0$, the velocity v and position s of the particle are known to be

$$v(0) = 3 \quad \text{m/sec, and} \qquad s(0) = 4 \text{ m.}$$

Find the position $s(t)$ of the particle for all $t > 0$.

To get velocity, take an anti derivative of acceleration

$$V(t) = 12\frac{t^{1+1}}{1+1} - 2t + C$$

$$= 6t^2 - 2t + C \qquad \text{impose } v(0) = 3$$

$$3 = V(0) = 6(0)^2 - 2(0) + C \implies C = 3$$

$v(t) = 6t^2 - 2t + 3$

To get s, take an antiderivative of v.

$$s(t) = 6\,\frac{t^{2+1}}{2+1} - 2\,\frac{t^{1+1}}{1+1} + 3t + C$$

$$s(t) = 2t^3 - t^2 + 3t + C$$

impose

$s(0) = 4$

$$4 = s(0) = 2(0)^3 - 0^2 + 3(0) + C \implies C = 4$$

So

$$\boxed{s(t) = 2t^3 - t^2 + 3t + 4, \quad t > 0}$$

Example

A **differential equation** is an equation that involves the derivative(s) of an unknown function. **Solving** such an equation would mean finding such an unknown function.

Solve the differential equation subject to the given *initial* conditions.

$$\frac{d^2y}{dx^2} = \cos x + 2, \quad y(0) = 0, \quad y'(0) = -1$$

Take an antiderivative of both sides

$$\frac{dy}{dx} = \sin x + 2x + C$$

Take another antiderivative

$$y = -\cos x + x^2 + Cx + K$$

Impose $y'(0) = -1$ and $y(0) = 0$

$$-1 = y'(0) = \sin(0) + 2(0) + C \Rightarrow C = -1$$

$$0 = y(0) = -\cos(0) + 0^2 - 1(0) + k$$

$$0 = -1 + k \Rightarrow k = 1$$

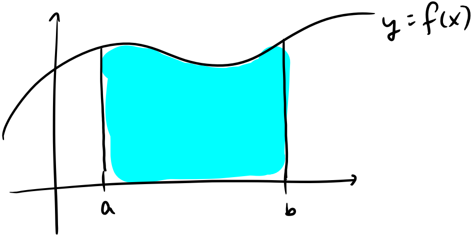
Hence

$$\boxed{y = -\cos x + x^2 - x + 1}$$

Section 4.1: Area Under a Positive Curve

Suppose f is a continuous function that is positive on the interval $a \leq x \leq b$. We can consider a region in the xy-plane bounded below by the x-axis, above by the curve $y = f(x)$ and on the sides by vertical segments of the lines $x = a$ and $x = b$.

Question: Can we find the area of such a region?



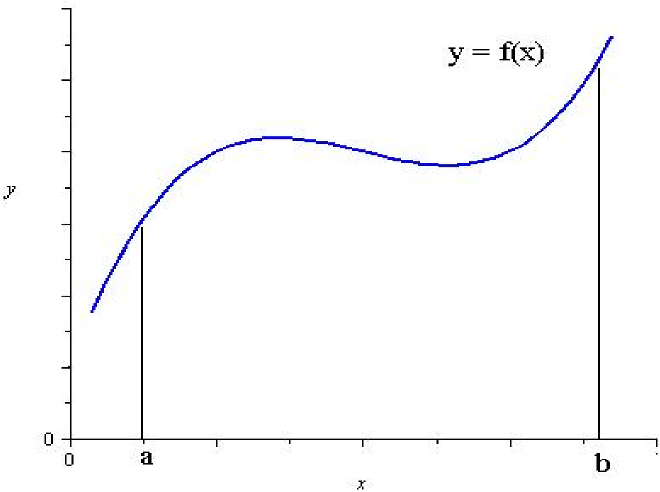


Figure: Region under a positive curve $y = f(x)$ on an interval $[a, b]$.

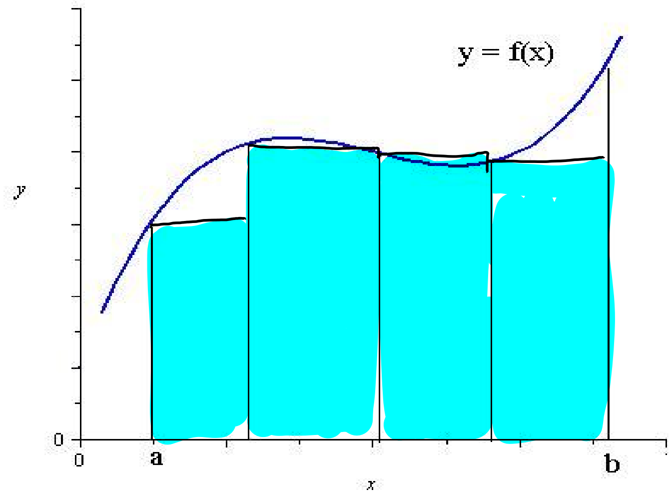
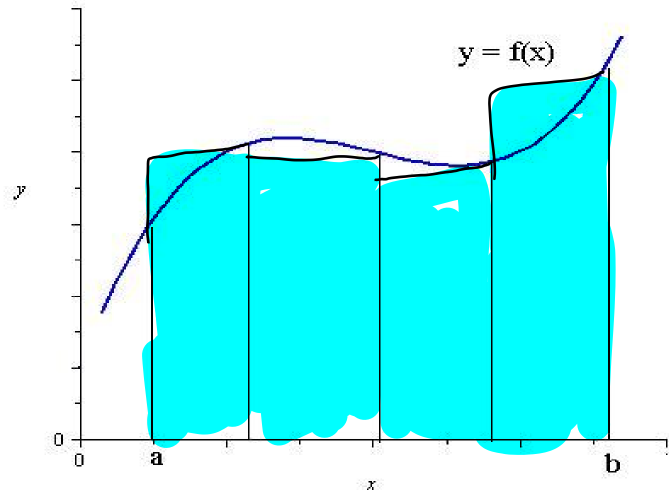


Figure: We could approximate the area by filling the space with rectangles.



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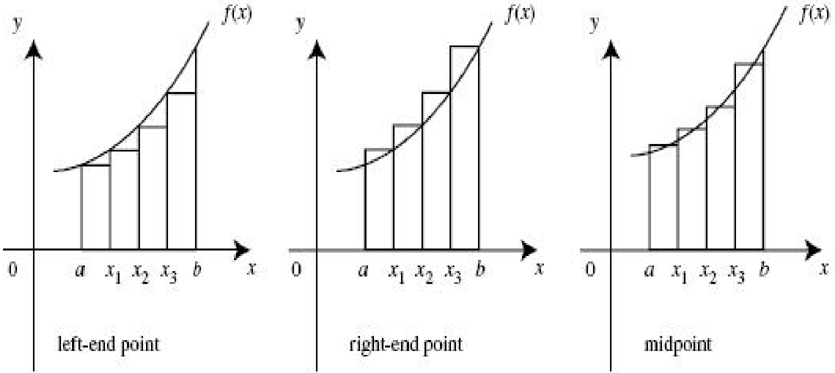


Figure: Some choices as to how to define the heights.

Approximating Area Using Rectangles

We can experiment with

- Which points to use for the heights (left, right, middle, other....)
- How many rectangles we use

to try to get a good approximation.

Definition: We will define the true area to be value we obtain taking the limit as the number of rectangles goes to $+\infty$.

Some terminology

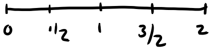
- A **Partition** P of an interval $[a, b]$ is a collection of points $\{x_0, x_1, ..., x_n\}$ such that
$$a = x_0 < x_1 < x_2 < \cdots < x_n = b.$$
- A **Subinterval** is one of the intervals $x_{i-1} \le x \le x_i$ determined by a partition.
- The width of a subinterval is denoted $\Delta x_i = x_i - x_{i-1}$. If they are all the same size (equal spacing), then
$$\Delta x = \frac{b-a}{n}, \quad \text{and this is called the } \textbf{norm} \text{ of the partition.}$$
- A set of **sample points** is a set $\{x_1^*, x_2^*, ..., x_n^*\}$ such that $x_{i-1} \le x_i^* \le x_i$.

Taking the number of rectangles to ∞ is the same as taking the width $\Delta x \to 0$.

Example: Write an equally spaced partition of the interval $[0, 2]$ with the specified number of subintervals, and determine the norm Δx.

(a) For $n = 4$

$$\Delta x = \frac{2-0}{4} = \frac{1}{2}$$

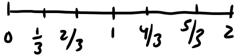


$x_0 = 0$, $x_1 = \frac{1}{2}$ $x_2 = 1$ $x_3 = \frac{3}{2}$ $x_4 = 2$

Note $x_i = x_0 + i\Delta x$, $i = 1, 2, 3, 4$

(b) For $n = 6$

$$\Delta x = \frac{2-0}{6} = \frac{1}{3}$$



$x_0 = 0$ $x_3 = 1$ $x_6 = 2$

$x_1 = \frac{1}{3}$ $x_4 = \frac{4}{3}$

$x_2 = \frac{2}{3}$ $x_5 = 5/3$

Again note that

$x_i = x_0 + i\Delta x$ for $i = 1, \ldots, 6$