Oct 16 Math 2253H sec. 05H Fall 2014

Section 4.1: Area Under a Positive Curve

Suppose f is a continuous function that is positive on the interval $a \leq x \leq b$. We can consider a region in the xy-plane bounded below by the x-axis, above by the curve $y = f(x)$ and on the sides by vertical segments of the lines $x = a$ and $x = b$.

Question: Can we find the area of such a region?

Approximating Area Using Rectangles

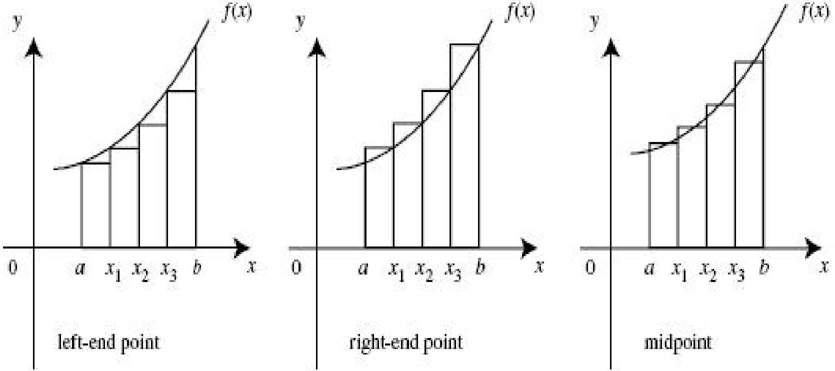


Figure: Some choices as to how to define the heights.

Approximating Area Using Rectangles

We can experiment with

- Which points to use for the heights (left, right, middle, other....)
- How many rectangles we use

to try to get a good approximation.

Definition: We will define the true area to be value we obtain taking the limit as the number of rectangles goes to $+\infty$.

Some terminology

- A **Partition** P of an interval $[a, b]$ is a collection of points $\{x_0, x_1, ..., x_n\}$ such that

$$a = x_0 < x_1 < x_2 < \cdots < x_n = b.$$

- A **Subinterval** is one of the intervals $x_{i-1} \le x \le x_i$ determined by a partition.
- The width of a subinterval is denoted $\Delta x_i = x_i - x_{i-1}$. If they are all the same size (equal spacing), then

$$\Delta x = \frac{b-a}{n},$$ and this is called the **norm** of the partition.

- A set of **sample points** is a set $\{x_1^*, x_2^*, ..., x_n^*\}$ such that $x_{i-1} \le x_i^* \le x_i$.

Taking the number of rectangles to ∞ is the same as taking the width $\Delta x \to 0$.

Example

We found that an equally spaced partition of $[0, 2]$ using $n = 6$ rectangles was

$$\left\{0, \frac{1}{3}, \frac{2}{3}, 1, \frac{4}{3}, \frac{5}{3}, 2\right\}$$

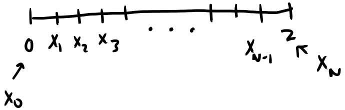
with $\Delta x = \frac{1}{3}$. We observed that

$$x_i = x_0 + i\Delta x, \quad \text{for} \quad i = 1, \ldots 6.$$

(c) Find an equally spaced partition of $[0, 2]$ having N subintervals. What is the norm Δx?

$a = 0, \quad b = 2$

$$\Delta x = \frac{2-0}{N} = \frac{2}{N}$$



$$x_0 = 0$$

$$x_1 = 0 + \Delta x = \frac{2}{N}$$

$$x_2 = x_1 + \Delta x = 2\Delta x = \frac{4}{N}$$

$$x_3 = x_2 + \Delta x = 3\Delta x = \frac{6}{N}$$

$$\vdots$$

$$x_i = 0 + i\,\Delta x = i\left(\frac{2}{N}\right) = \frac{2i}{N}$$

$$x_N = \frac{2N}{N} = 2$$

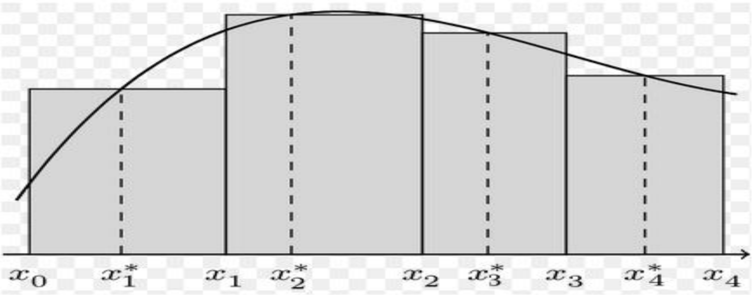
Partition:

$$\left\{0, \frac{2}{N}, \frac{4}{N}, \ldots, 2\right\}$$

$$\left\{x_i \mid x_i = i\,\frac{2}{N}, \; i = 0, \ldots, N\right\}$$

Norm $\quad \Delta x = \dfrac{2}{N}$

Approximating area with a Partition and sample points



height

width

Figure: Area = $f(x_1^*)\Delta x + f(x_2^*)\Delta x + f(x_3^*)\Delta x + f(x_4^*)\Delta x$. This can be written as

$$\sum_{i=1}^{n} f(x_i^*)\Delta x.$$

Sigma Notation

Suppose that $a_1, a_2, \ldots, a_n$ are a collection of real numbers. Then

$$\sum_{i=1}^{n} a_i = a_1 + a_2 + \cdots + a_n.$$

Capital sigma — index — first index value — last index value

Reads as

the sum from $i = 1$ *to* n *of* a *sub* i.

In general, an equally spaced partition of $[a, b]$ with n subintervals means

- $\Delta x = \frac{b-a}{n}$
- $x_0 = a$, $x_1 = a + \Delta x$, $x_2 = a + 2\Delta x$, i.e. $x_i = a + i\Delta x$
- Taking heights to be

$$\text{left ends} \quad x_i^* = x_{i-1} \quad \text{area} \approx \sum_{i=1}^{n} f(x_{i-1})\Delta x$$

$$\text{right ends} \quad x_i^* = x_i \quad \text{area} \approx \sum_{i=1}^{n} f(x_i)\Delta x$$

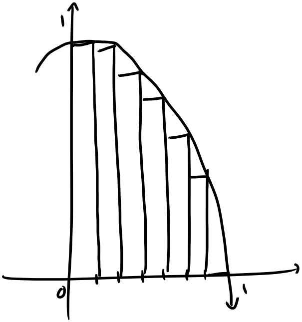
- The true area exists (for f continuous) and is given by

$$\lim_{n\to\infty} \sum_{i=1}^{n} f(x_i^*)\Delta x.$$

Example: Find the area under the curve $y = 1 - x^2$, $0 \leq x \leq 1$.

Use right end points $x_i^* = x_i$ and assume the following identity

$$\sum_{i=1}^{n} i^2 = \frac{2n^3 + 3n^2 + n}{6} \quad \text{(sum of first } n \text{ squares)}$$



$$\Delta x = \frac{b-a}{n} = \frac{1-0}{n} = \frac{1}{n}$$

$$x_0 = 0$$

$$x_i = x_0 + i\Delta x = 0 + i\left(\frac{1}{n}\right) = \frac{i}{n}$$

$$x_i = \frac{i}{n}$$

$$x_i^* = x_i = \frac{i}{n}$$

$$f(x) = 1 - x^2 \Rightarrow f(x_i) = 1 - (x_i)^2 = 1 - \frac{i^2}{n^2}$$

$$\text{area} \approx \sum_{i=1}^{n} f(x_i)\Delta x = \sum_{i=1}^{n} \left(1 - \frac{i^2}{n^2}\right)\frac{1}{n}$$

$$= \frac{1}{n}\sum_{i=1}^{n}\left(1 - \frac{i^2}{n^2}\right)$$

$$= \frac{1}{n}\sum_{i=1}^{n} 1 - \frac{1}{n}\sum_{i=1}^{n}\frac{i^2}{n^2}$$

$$= \frac{1}{n}\sum_{i=1}^{n} 1 - \frac{1}{n^3}\sum_{i=1}^{n} i^2$$

$$\frac{i^2}{n^2} = \frac{1}{n^2}\, i^2$$

$$\sum_{i=1}^{n} 1 = \underbrace{1 + 1 + 1 + \dots + 1}_{n \text{ of these}} = n$$

$$\text{area} \approx \frac{1}{n}(n) - \frac{1}{n^3}\left(\frac{2n^3 + 3n^2 + n}{6}\right)$$

$$= 1 - \frac{2n^3 + 3n^2 + n}{6n^3}$$

To find the exact area, take $n \to \infty$.

$$\text{Area} = \lim_{n\to\infty}\left(1 - \frac{2n^3+3n^2+n}{6n^3}\right)$$

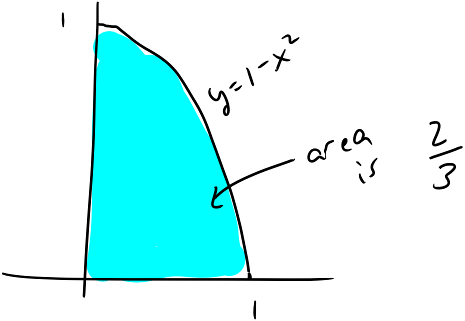
$$= \lim_{n\to\infty} 1 - \lim_{n\to\infty}\frac{2n^3+3n^2+n}{6n^3}$$

$$= 1 - \lim_{n\to\infty}\left(\frac{2n^3+3n^2+n}{6n^3}\right)\cdot\frac{\frac{1}{n^3}}{\frac{1}{n^3}}$$

$$= 1 - \lim_{n\to\infty}\frac{2+\frac{3}{n}+\frac{1}{n^2}}{6}$$

$$= 1 - \frac{2+0+0}{6} = 1 - \frac{1}{3} = \frac{2}{3}$$

The area under the curve is $\frac{2}{3}$ square units.



Recovering Distance from Velocity

The speedometer readings for a motorcycle are recorded at 12 second intervals. Use the information in the table to estimate the total distance traveled. Get estimates using
(a) left end points (beginning time of intervals), and
(b) right end points (ending time for each interval).

t in sec	0	12	24	36	48	60
v in ft/sec	20	28	25	22	24	27

t in sec	0	12	24	36	48	60
v in ft/sec	20	28	25	22	24	27

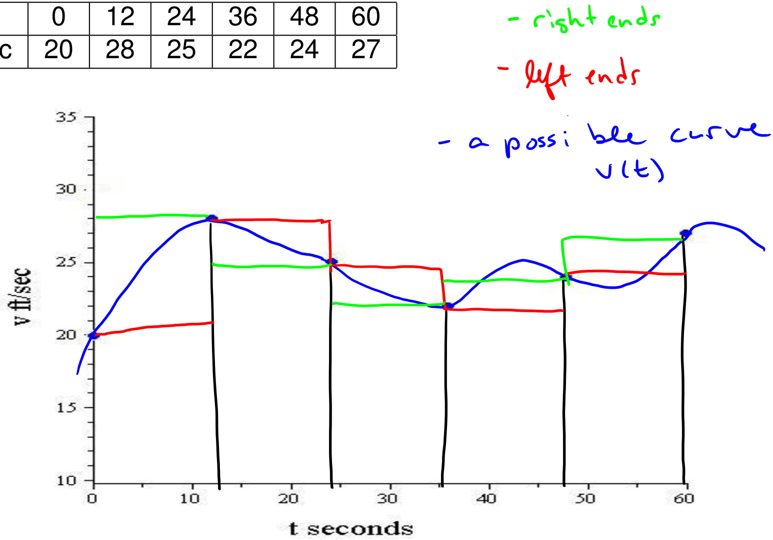


Figure: Graphical representation of motorcycle's velocity.

t in sec	0	12	24	36	48	60
v in ft/sec	20	28	25	22	24	27

Using left: Distance D

$$D \approx 20\,\frac{\text{ft}}{\text{sec}}\cdot 12\text{ sec} + 28\,\frac{\text{ft}}{\text{sec}}\cdot 12\text{ sec} + 25\,\frac{\text{ft}}{\text{sec}}\cdot 12\text{ sec}$$

$$+\, 22\,\frac{\text{ft}}{\text{sec}}\cdot 12\text{ sec.} + 24\,\frac{\text{ft}}{\text{sec}}\cdot 12\text{ sec.}$$

$$D \approx 1428\text{ ft}$$

t in sec	0	12	24	36	48	60
v in ft/sec	20	28	25	22	24	27

Right ends:

$$D \approx 28\,\frac{\text{ft}}{\text{sec}}\cdot 12\text{ sec} + 25\,\frac{\text{ft}}{\text{sec}}\cdot 12\text{ sec} + 22\,\frac{\text{ft}}{\text{sec}}\cdot 12\text{ sec}$$

$$+ 24\,\frac{\text{ft}}{\text{sec}}\cdot 12\text{ sec} + 27\,\frac{\text{ft}}{\text{sec}}\cdot 12\text{ sec}$$

$$D \approx 1512\text{ ft}$$