Oct 17 Math 2253H sec. 05H Fall 2014

Section 4.2: The Definite Integral

We saw that a sum of the form

$$f(x_1^*)\Delta x + f(x_2^*)\Delta x + \cdots + f(x_n^*)\Delta x$$

approximated the area of a region if f was continuous and positive. And that under these conditions, the limit

$$\lim_{n\to\infty}\sum_{i=1}^{n} f(x_i^*)\Delta x = \lim_{n\to\infty}[f(x_1^*)\Delta x + f(x_2^*)\Delta x + \cdots + f(x_n^*)\Delta x]$$

was the value of this area.

Can we generalize this dropping the requirement that f is positive? that f is continuous?

Definition (Definite Integral)

Let f be defined on an interval $[a, b]$. Let

$$x_0 = a < x_1 < x_2 < \cdots < x_n = b$$

be any partition of $[a, b]$, and $\{x_1^*, x_2^*, \dots, x_n^*\}$ be any set of sample points. Then the **definite integral of** f **from** a **to** b is denoted and defined by

$$\int_a^b f(x)\,dx = \lim_{n\to\infty} \sum_{i=1}^{n} f(x_i^*)\Delta x_i$$

provided this limit exists. Here, the limit is taken over all possible partitions of $[a, b]$.

Terms and Notation

- **Riemann Sum:** any sum of the form $f(x_1^*)\Delta x + f(x_2^*)\Delta x + \cdots + f(x_n^*)\Delta x$
- **Integral Symbol/Sign:** $\int$ (a stretched "S" for "sum")
- **Integrable:** If the limit does exists, f is said to be integrable on $[a, b]$
- **Limits of Integration:** a is called the lower limit of integration, and b is the upper limit of integration
- **Integrand:** the expression "$f(x)$" is called the integrand

- **Differential:** dx is called a differential, it indicates what the variable is and can be thought of as the limit of Δx (just as it is in the derivative notation " $\frac{dy}{dx}$ ").

- **Dummy Variable/Variable of Integration:** the variable that appears in both the integrand and in the differential. For example, if the differential is dx, the dummy variable is x; it the differential is du, the dummy variable is u

$$\int_a^b f(x)\,dx$$

Important Remarks

(1) If the integral does exist, it is a **number**. That is, it does not depend on the dummy variable of integration. The following are equivalent.

$$\int_a^b f(x)\,dx = \int_a^b f(t)\,dt = \int_a^b f(q)\,dq$$

(2) The definite integral is a limit of Riemann Sums!

(3) If f is positive and continuous on $[a, b]$, then

$$\int_a^b f(x)\,dx = \text{ the area under the curve.}$$

What if f is continuous, but not always positive?

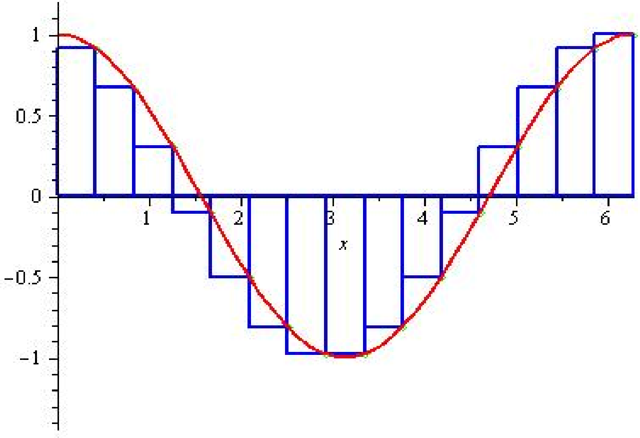


Figure: A function that changes signs on $[a, b]$. (Here, $f(x) = \cos x$, $a = 0$ and $b = 2\pi$; the partition has 15 subintervals.)

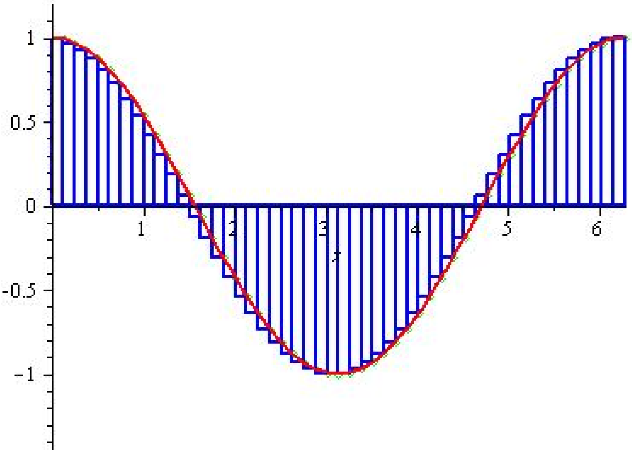


Figure: The same function but with 50 subintervals.

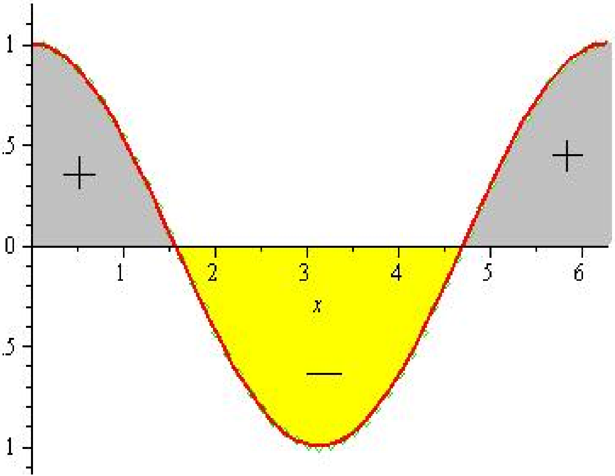
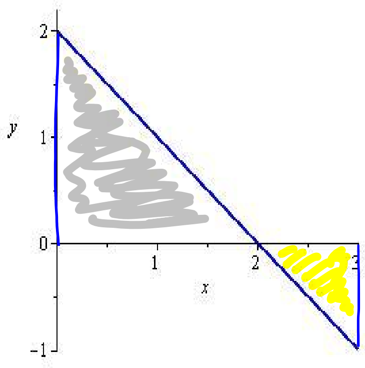


Figure: $\int_a^b f(x)\,dx$ = area of gray region − area of yellow region

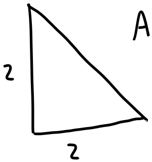
Example

Use area to evaluate the integral $\int_0^3 (2-x)\,dx$.



$\int_0^3 (2-x)\,dx$ = gray area − yellow area

gray part

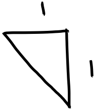


$A = \frac{1}{2} bh$

$= \frac{1}{2}(2)(2)$

$= 2$

yellow part



$A = \frac{1}{2} bh$

$= \frac{1}{2}(1)(1) = \frac{1}{2}$

$$\int_0^3 (2-x)\,dx = 2 - \frac{1}{2} = \frac{3}{2}$$