Oct 24 Math 2253H sec. 05H Fall 2014

Section 4.2: The Definite Integral

Properties of Definite Integrals

Suppose that f and g are integable on $[a, b]$ and let c be constant.

(1) $$\int_a^b c\,dx = c(b-a)$$

(2) $$\int_a^b cf(x)\,dx = c\int_a^b f(x)\,dx$$

(3) $$\int_a^b (f(x) \pm g(x))\,dx = \int_a^b f(x)\,dx \pm \int_a^b g(x)\,dx$$

Properties of Definite Integrals Continued...

(4) $\displaystyle\int_b^a f(x)\,dx = -\int_a^b f(x)\,dx$

(5) $\displaystyle\int_a^a f(x)\,dx = 0$

(6) $\displaystyle\int_a^b f(x)\,dx = \int_a^c f(x)\,dx + \int_c^b f(x)\,dx$

Properties of Definite Integrals Continued...

(7) If $f(x) \leq g(x)$ for $a \leq x \leq b$, then $\displaystyle\int_a^b f(x)\,dx \leq \int_a^b g(x)\,dx$

(8) And, as an immediate consequence of (7) and (1), if $m \leq f(x) \leq M$ for $a \leq x \leq b$, then

$$m(b-a) \leq \int_a^b f(x)\,dx \leq M(b-a).$$

Example Problems:

(1) Evaluate the integral using areas and the properties.

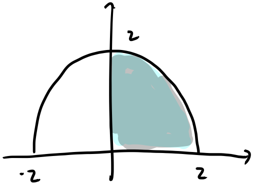
$$\int_2^0 \sqrt{4 - x^2}\, dx$$

$f(x) = \sqrt{4-x^2}$ $\quad y = \sqrt{4-x^2}$

$y^2 = 4 - x^2 \Rightarrow x^2 + y^2 = 4$

$= -\int_0^2 \sqrt{4-x^2}\, dx$

property (4)



Shaded area $= \int_0^2 \sqrt{4-x^2}\, dx$

Shaded area $= \frac{1}{4}\pi(2)^2$

$= \pi$

So $$\int_2^0 \sqrt{4-x^2}\,dx = -\pi$$

(2) Use the given properties to argue that

$$6 \leq \int_0^3 \sqrt{4+t^2}\, dt \leq 3\sqrt{13}.$$

property (8) if $m \leq f(x) \leq M$ on $[a,b]$ then

$$m(b-a) \leq \int_a^b f(x)\,dx \leq M(b-a)$$

$f(t) = \sqrt{4+t^2}$ for $0 \leq t \leq 3$

Look for abs. max and min. Find crit. #

$$f'(t) = \frac{1}{\cancel{2}}(4+t^2)^{-1/2} \cdot (\cancel{2}t) = \frac{t}{\sqrt{4+t^2}}$$

$f'(t) = 0 \Rightarrow t = 0$, $f'(t)$ undefined $\sqrt{4+t^2} = 0$

no soln.

$f(0) = \sqrt{4+0^2} = \sqrt{4} = 2$ ← abs min

$f(3) = \sqrt{4+3^2} = \sqrt{13}$ ← abs max

$$2 \le f(t) \le \sqrt{13} \quad \text{on} \quad [0,3]$$

So $$2(3-0) \le \int_0^3 \sqrt{4+t^2}\,dt \le \sqrt{13}\,(3-0)$$

$$6 \le \int_0^3 \sqrt{4+t^2}\,dt \le 3\sqrt{13}$$

(3) Given $\int_0^5 f(x)\,dx = 7$, $\int_2^5 f(x)\,dx = -3$, and $\int_0^5 g(x)\,dx = 3$, evaluate

(a) $\displaystyle\int_0^2 f(x)\,dx$

By property (6)

$$\int_0^5 f(x)\,dx = \int_0^2 f(x)\,dx + \int_2^5 f(x)\,dx$$

$$7 = \int_0^2 f(x)\,dx + (-3) \Rightarrow \int_0^2 f(x)\,dx = 7+3 = 10$$

(b) $\displaystyle\int_0^5 (2g(x) - 3f(x))\,dx$

$$= \int_0^5 2g(x)\,dx - \int_0^5 3f(x)\,dx$$

property (3)

$$= 2\int_0^5 g(x)\,dx - 3\int_0^5 f(x)\,dx$$

property (2)

$$= 2(3) - 3(7) = -15$$

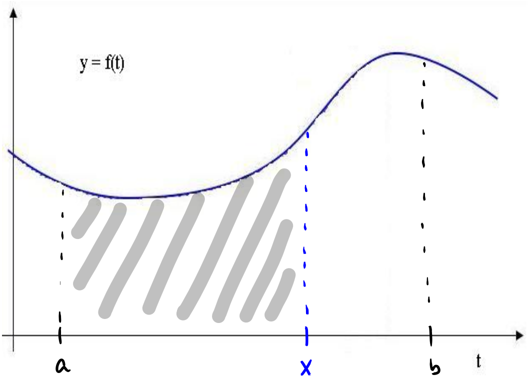
Section 4.3: The Fundamental Theorem of Calculus

Suppose f is continuous on the interval $[a, b]$. For $a \leq x \leq b$ define a new function

$$g(x) = \int_a^x f(t)\,dt$$

How can we understand this function, and what can be said about it?

Geometric interpretation of $g(x) = \int_a^x f(t)\,dt$



Figure

g(x) would correspond to the area under f between t=a and t=x

$g(a) = 0$

FTC Applet 1 FTC Applet 2

Theorem: The Fundamental Theorem of Calculus (part 1)

If f is continuous on $[a, b]$ and the function g is defined by

$$g(x) = \int_a^x f(t)\,dt \quad \text{for} \quad a \le x \le b,$$

then g is continuous on $[a, b]$ and differentiable on (a, b). Moreover

$$g'(x) = f(x).$$

This means that the new function g is an **antiderivative** of f on (a, b)!
”FTC” = ”fundamental theorem of calculus”

Example:

Evaluate each derivative.

$f(t) = \sin^2(t)$

$g(x) = \int_0^x f(t)\,dt$

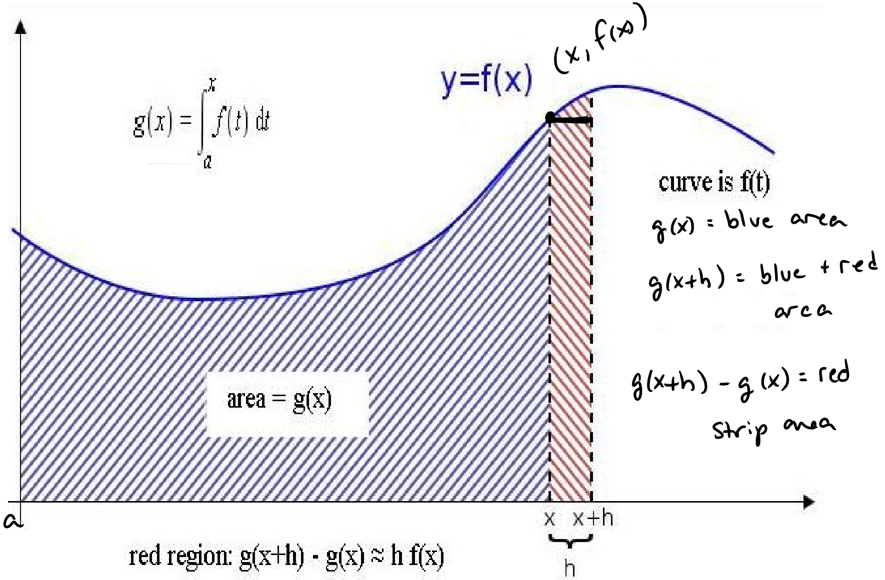
(a) $\frac{d}{dx}\int_0^x \sin^2(t)\,dt = \sin^2(x)$

$f(t) = \frac{t - \cos t}{t^4 + 1}$

(b) $\frac{d}{dx}\int_4^x \frac{t - \cos t}{t^4 + 1}\,dt = \frac{x - \cos x}{x^4 + 1}$

$g(x) = \int_4^x f(t)\,dt$

Geometric Argument of FTC



If it exists $g'(x) = \lim_{h \to 0} \dfrac{g(x+h) - g(x)}{h}$

h = width red region $\qquad f(x)$ = height of rectangle

$$g(x+h) - g(x) \approx h\,f(x) \quad \Rightarrow$$

$$\frac{g(x+h) - g(x)}{h} \approx f(x) \quad \Rightarrow$$

$$\lim_{h \to 0} \frac{g(x+h) - g(x)}{h} = f(x) = g'(x)$$