Oct 27 Math 2253H sec. 05H Fall 2014

Section 4.3: The Fundamental Theorem of Calculus

Theorem: The Fundamental Theorem of Calculus (part 1) If f is continuous on $[a, b]$ and the function g is defined by

$$g(x) = \int_a^x f(t)\,dt \quad \text{for} \quad a \le x \le b,$$

then g is continuous on $[a, b]$ and differentiable on (a, b). Moreover

$$g'(x) = f(x).$$

This means that the new function g is an **antiderivative** of f on (a, b)!
"FTC" = "fundamental theorem of calculus"

Recall our Examples

Evaluate each derivative.

(a) $$\frac{d}{dx}\int_0^x \sin^2(t)\,dt = \sin^2(x)$$

(b) $$\frac{d}{dx}\int_4^x \frac{t-\cos t}{t^4+1}\,dt = \frac{x-\cos x}{x^4+1}$$

Chain Rule with FTC

Evaluate each derivative.

(a) $\displaystyle \frac{d}{dx}\int_0^{x^2} t^3\,dt$

$= (x^2)^3(2x)$

$= x^6(2x)$

$= 2x^7$

Chain rule: If $u = g(x)$

and $y = f(u) = f(g(x))$

then

$$\frac{dy}{dx} = \frac{dy}{du}\cdot\frac{du}{dx}$$

$$= f'(g(x))\,g'(x)$$

Here, $u = x^2$ and

$$f(u) = \int_0^u t^3\,dt$$

$f'(u) = u^3$ and $u'(x) = 2x$

(b) $\dfrac{d}{dx}\displaystyle\int_x^7 \cos(t^2)\,dt$

Property of integrals:

$$\int_b^a f(t)\,dt = -\int_a^b f(t)\,dt$$

$$= \frac{d}{dx}\left(-\int_7^x \cos(t^2)\,dt\right)$$

$$= -\cos(x^2)\cdot 1$$

$$= -\cos(x^2)$$

Leibniz Rule

Suppose a and b are differentiable functions and f is continuous.

$$\frac{d}{dx}\int_{a(x)}^{b(x)} f(t)\,dt = f(b(x))b'(x) - f(a(x))a'(x)$$

Example:

$$\frac{d}{dt}\int_{x^2}^{\sqrt{x}} f(t)\,dt = f(\sqrt{x})\left(\frac{1}{2\sqrt{x}}\right) - f(x^2)(2x) = \frac{f(\sqrt{x})}{2\sqrt{x}} - 2xf(x^2).$$

$\frac{d}{dx}\sqrt{x}$ $\frac{d}{dx}x^2$

Example

Evaluate the derivative $\displaystyle \frac{d}{dx}\int_{\sin x}^{\cos x} 3t^2\,dt$

$a(x) = \sin x$

$b(x) = \cos x$

$f(t) = 3t^2$

$$= \underbrace{3(\cos x)^2}_{f(\cos x)}\underbrace{(-\sin x)}_{(\cos x)'} - \underbrace{3(\sin x)^2}_{f(\sin x)}\underbrace{(\cos x)}_{(\sin x)'}$$

$$= -3\cos^2 x\,\sin x - 3\sin^2 x\,\cos x$$

Theorem: The Fundamental Theorem of Calculus (part 2)

If f is continuous on $[a, b]$, then

$$\int_a^b f(x)\,dx = F(b) - F(a)$$

where F is any antiderivative of f on $[a, b]$. (i.e. $F'(x) = f(x)$)

To evaluate $\int_a^b f(x)dx$, find an anti derivative $F(x)$

evaluate $F(b)$ and $F(a)$, compute the difference

$F(b) - F(a)$.

Example: Use the FTC to show that $\int_0^b x\,dx = \frac{b^2}{2}$

power rule for anti derivatives

$x^n \rightarrow \frac{x^{n+1}}{n+1}, \; n \neq -1$

$f(x) = x$

$F(x) = \frac{x^{1+1}}{1+1} = \frac{x^2}{2}$

so

$$\int_0^b x\,dx = F(b) - F(0)$$

$$= \frac{b^2}{2} - \frac{0^2}{2} = \frac{b^2}{2}$$

Notation

Suppose F is an antiderivative of f. We write

$$\int_a^b f(x)\,dx = F(x)\big|_a^b = F(b) - F(a)$$

or sometimes

$$\int_a^b f(x)\,dx = F(x)\big]_a^b = F(b) - F(a)$$

For example

$$\int_0^b x \;\, dx = \left.\frac{x^2}{2}\right|_0^b = \frac{b^2}{2} - \frac{0^2}{2} = \frac{b^2}{2}$$

Evaluate each definite integral using the FTC

(a) $\int_0^2 3x^2\,dx = x^3\Big|_0^2 = 2^3 - 0^3$

$= 8 - 0$

$= 8$

(b) $\int_{\frac{\pi}{2}}^{\pi} \cos x \, dx = \sin x + 1 \Big|_{\pi/2}^{\pi}$

$$= \sin \pi + 1 - \left(\sin \tfrac{\pi}{2} + 1 \right)$$

$$= \sin \pi + \cancel{1} - \sin \tfrac{\pi}{2} - \cancel{1}$$

$$= 0 - 1$$

$$= -1$$

(c) $\int_{-1}^{2}(y^2+2)\,dy = \int_{-1}^{2} y^2\,dy + \int_{-1}^{2} 2\,dy$

$$= \frac{y^3}{3}\Big|_{-1}^{2} + 2y\Big|_{-1}^{2}$$

$$= \frac{2^3}{3} - \frac{(-1)^3}{3} + \left(2\cdot 2 - 2\cdot(-1)\right)$$

$$= \frac{8}{3} + \frac{1}{3} + 4 + 2 = \frac{9}{3} + 6 = 9$$

(d) $\int_1^8 \sqrt[3]{u}\, du = \int_1^8 u^{1/3}\, du$

$\frac{1}{3} + 1 = \frac{4}{3}$

$= \left. \frac{u^{4/3}}{4/3} \right|_1^8$

$= \left. \frac{3}{4} u^{4/3} \right|_1^8 = \frac{3}{4}(8)^{4/3} - \frac{3}{4}(1)^{4/3}$

$= \frac{3}{4}(16) - \frac{3}{4}(1) = \frac{3}{4}(15) = \frac{45}{4}$

(e) $\int_0^2 (t+3)(2t-1)\,dt = \int_0^2 (2t^2 + 5t - 3)\,dt$

$$= \left[2\frac{t^3}{3} + 5\frac{t^2}{2} - 3t\right|_0^2$$

$$= \left(\frac{2}{3}\cdot 2^3 + \frac{5}{2}\cdot 2^2 - 3\cdot 2\right) - \left(\frac{2}{3}\cdot 0^3 + \frac{5}{2}\cdot 0^2 - 3\cdot 0\right)$$

$$= \frac{16}{3} + 10 - 6 = \frac{16}{3} + 4 = \frac{16}{3} + \frac{12}{3} = \frac{28}{3}$$

Example

Use the second FTC to evaluate the derivative. Compare this to the result using the first FTC with the chain rule.

$$\frac{d}{dx}\int_{\sin x}^{\cos x} 3t^2\,dt$$

We'll evaluate $\int_{\sin x}^{\cos x} 3t^2\,dt$ first, then take the derivative.

$$\int_{\sin x}^{\cos x} 3t^2\,dt = t^3\Big|_{\sin x}^{\cos x} = (\cos x)^3 - (\sin x)^3$$

So $$\frac{d}{dx}\int_{\sin x}^{\cos x} 3t^2\,dx = \frac{d}{dx}\left((\cos x)^3 - (\sin x)^3\right)$$

$$= 3(\cos x)^2 \cdot (-\sin x) - 3(\sin x)^2 \cdot (\cos x)$$

$$= -3\cos^2 x \sin x - 3\sin^2 x \cos x$$

Caveat! The FTC doesn't apply if f is not continuous!

The function $f(x) = \frac{1}{x^2}$ is positive everywhere on its domain. Now consider the calculation

$$\int_{-1}^{2} \frac{1}{x^2}\,dx = \left.\frac{x^{-1}}{-1}\right|_{-1}^{2} = -\frac{1}{2} - 1 = -\frac{3}{2}$$

a positive function can't give a negative integral!

Is this believable? Why or why not?

The FTC doesn't apply since f is not continuous on $[-1, 2]$. The integral $\int_{-1}^{2} \frac{1}{x^2}\,dx$ is undefined.