Oct 28 Math 2253H sec. 05H Fall 2014

Section 4.3: The Fundamental Theorem of Calculus (FTC)

Theorem: The Fundamental Theorem of Calculus (part 1) If f is continuous on $[a, b]$ and the function g is defined by

$$g(x) = \int_a^x f(t)\,dt \quad \text{for} \quad a \le x \le b,$$

then g is continuous on $[a, b]$ and differentiable on (a, b). Moreover

$$g'(x) = f(x).$$

This means that the new function g is an **antiderivative** of f on (a, b)!

Theorem: The Fundamental Theorem of Calculus (part 2)

If f is continuous on $[a, b]$, then

$$\int_a^b f(x)\,dx = F(x)\Big|_a^b = F(b) - F(a)$$

where F is any antiderivative of f on $[a, b]$. (i.e. $F'(x) = f(x)$)

To evaluate $\int\limits_a^b f(x)\,dx$, we

- find any antiderivative F of f,
- evaluate $F(b)$ and $F(a)$, and
- take the difference $F(b) - F(a)$.

Evaluate each derivative or definite integral using the FTC (part 1 or 2)

(a) $\displaystyle\int_1^5 \frac{1}{x^2}\,dx \quad = \int_1^5 x^{-2}\,dx$

$\displaystyle = \frac{x^{-1}}{-1}\Big|_1^5 = \frac{-1}{x}\Big|_1^5 = \frac{-1}{5} - \frac{-1}{1} = 1 - \frac{1}{5} = \frac{4}{5}$

(b) $\int_0^{\pi/4} \sec^2 x \, dx = \tan x \Big|_0^{\pi/4}$

$= \tan\frac{\pi}{4} - \tan 0 = 1 - 0 = 1$

(c) $\frac{d}{dx}\int_1^{x^4} \frac{1}{t^2+1}\,dt = \frac{4x^3}{x^8+1}$

for $f(u) = \int_1^u \frac{1}{t^2+1}\,dt \quad f'(u) = \frac{1}{u^2+1}$

$u = g(x) = x^4, \quad g'(x) = 4x^2$

$\frac{d}{dx} f(g(x)) = f'(g(x))\,g'(x) = \frac{1}{(x^4)^2+1} \cdot 4x^3$

(d) $\int_{\pi/4}^{\pi/2} (\csc x \cot x + 4)\, dx$

$$= -\csc x + 4x \Big|_{\pi/4}^{\pi/2}$$

$$= -\csc\frac{\pi}{2} + 4\cdot\frac{\pi}{2} - \left(-\csc\frac{\pi}{4} + 4\cdot\frac{\pi}{4}\right)$$

$$= -1 + 2\pi - (-\sqrt{2} + \pi)$$

$$= -1 + 2\pi + \sqrt{2} - \pi \qquad = \sqrt{2} + \pi - 1$$

(e) $\frac{d}{dx}\int_x^5 (t+3)(2t-1)\,dt$

$$= \frac{d}{dx} - \int_5^x (t+3)(2t-1)\,dt$$

$$= -(x+3)(2x-1)$$

$$= -2x^2 - 5x + 3$$

(f) $\displaystyle\int_1^4 \frac{t^3+\sqrt{t}}{t^2}\,dt = \int_1^4 \left(t + t^{-3/2}\right)dt$

$$= \frac{t^2}{2} - \frac{2}{\sqrt{t}}\Bigg|_1^4$$

$$= (8-1) - \left(\tfrac{1}{2} - 2\right) = \frac{17}{2}$$

Section 4.4: Indefinite Integrals and Net Change

New notation for antiderivatives:

If $F'(x) = f(x)$, i.e. F is any antiderivative of f, we will write

$$\int f(x)\,dx = F(x) + C$$

and we'll call $\int f(x)\,dx$ the **indefinite** integral of f.

For example:

$$\int 2x\,dx = x^2 + C, \quad \text{and} \quad \int \cos t\,dt = \sin t + C$$

Note:

$$\int_a^b f(x)\,dx$$

is called the "definite integral of f from a to b." And, it is a number.

$$\int f(x)\,dx$$

is called an "indefinite integral of f". And, it is a family of functions.

Table of Indefinite Integrals (things we already know)

$$\int cf(x)\,dx = c\int f(x)\,dx$$

$$\int (f(x) \pm g(x))\,dx = \int f(x)\,dx \pm \int g(x)\,dx$$

$$\int k\,dx = kx + C$$

$$\int x^n\,dx = \frac{x^{n+1}}{n+1} + C, \quad \text{for } n \neq -1$$

Table Continued...

$$\int \sin x\, dx = -\cos x + C, \qquad \int \cos x\, dx = \sin x + C$$

$$\int \sec^2 x\, dx = \tan x + C, \qquad \int \csc^2 x\, dx = -\cot x + C$$

$$\int \sec x \tan x\, dx = \sec x + C, \qquad \int \csc x \cot x\, dx = -\csc x + C$$

Evaluate Each Integral

(a) $\int (t^3 - 2t + \sqrt{t})\,dt$ $= \int (t^3 - 2t + t^{1/2})\,dt$

$$= \frac{t^4}{4} - 2\frac{t^2}{2} + \frac{t^{3/2}}{3/2} + C$$

$$= \frac{t^4}{4} - t^2 + \frac{2}{3}t^{3/2} + C$$

(b) $\int_0^2 (x+1)^2\,dx = \int_0^2 (x^2+2x+1)\,dx$

$$= \frac{x^3}{3} + x^2 + x \Big|_0^2$$

$$= \frac{2^3}{3} + 2^2 + 2 - \left(\frac{0^3}{3} + 0^2 + 0\right)$$

$$= \frac{8}{3} + 4 + 2 = \frac{8}{3} + 6 = \frac{8+18}{3} = \frac{26}{3}$$

(c) $\int \tan^2\theta \, d\theta$

$= \int (\sec^2\theta - 1) \, d\theta$

$= \tan\theta - \theta + C$

Observations

$\tan^2\theta = \tan\theta \tan\theta$

$= \frac{\sin^2\theta}{\cos^2\theta}$

$= \frac{\sec^2\theta}{\csc^2\theta}$

$\tan^2\theta + 1 = \sec^2\theta$

$\tan^2\theta = \sec^2\theta - 1$

(d) $\int_{\pi/4}^{\pi/2} \frac{dx}{\sin^2 x} \quad = \quad \int_{\pi/4}^{\pi/2} \csc^2 x \, dx$

$$= -\cot x \Big|_{\pi/4}^{\pi/2}$$

$$= -\cot \frac{\pi}{2} - \left(-\cot \frac{\pi}{4}\right)$$

$$= -0 - (-1) = 1$$

(e) $\int \frac{x-1}{\sqrt[3]{x^2}}\,dx = \int \frac{x-1}{x^{2/3}}\,dx$

$$= \int \left(x^{1/3} - x^{-2/3} \right) dx$$

$$= \frac{x^{4/3}}{4/3} - \frac{x^{1/3}}{1/3} + C$$

$$= \frac{3x^{4/3}}{4} - 3\sqrt[3]{x} + C$$