Oct 2 Math 2253H sec. 05H Fall 2014

Section 3.3: Derivatives and the Shapes of Graphs

Find all the critical points of the function and classify each one as a local maximum, a local minimum, or neither.

$$f(x) = x^{1/3}(16 - x) = 16x^{1/3} - x^{4/3}$$

$$f'(x) = 16 \cdot \frac{1}{3}x^{-2/3} - \frac{4}{3}x^{1/3}$$

$$= \frac{16}{3x^{2/3}} - \frac{4x^{1/3}}{3} \cdot \frac{x^{2/3}}{x^{2/3}}$$

$$= \frac{16}{3x^{2/3}} - \frac{4x}{3x^{2/3}}$$

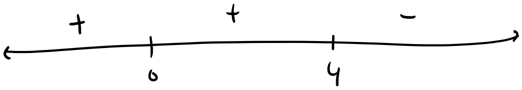
$$f'(x) = \frac{16 - 4x}{3x^{2/3}}$$

$f'(x) = 0 \Rightarrow 16 - 4x = 0 \Rightarrow x = 4$

$f'(x)$ DNE $\Rightarrow 3x^{2/3} = 0 \Rightarrow x = 0$

$$f'(x) = \frac{4(4-x)}{3\sqrt[3]{x^2}}$$

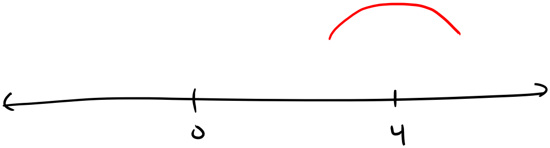
2 critical numbers 0 and 4



$f'(-1)$ $\frac{+}{+}$ $\quad f'(1)$ $\frac{+}{+}$ $\quad f'(5)$ $\frac{-}{+}$

f takes neither a local max or min @ $x = 0$

f takes a local maximum at $x=4$.



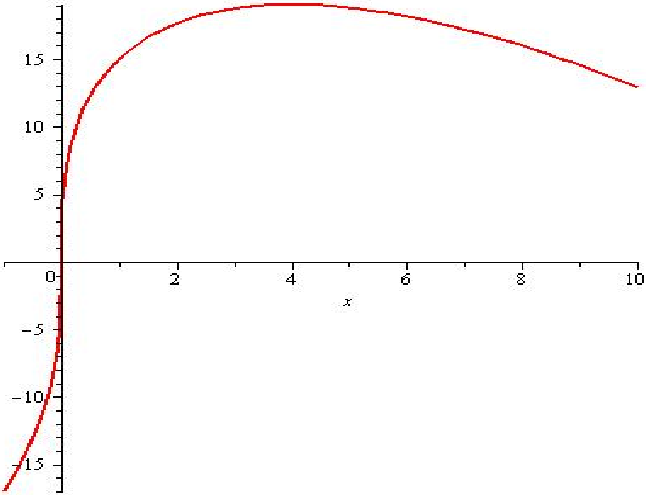
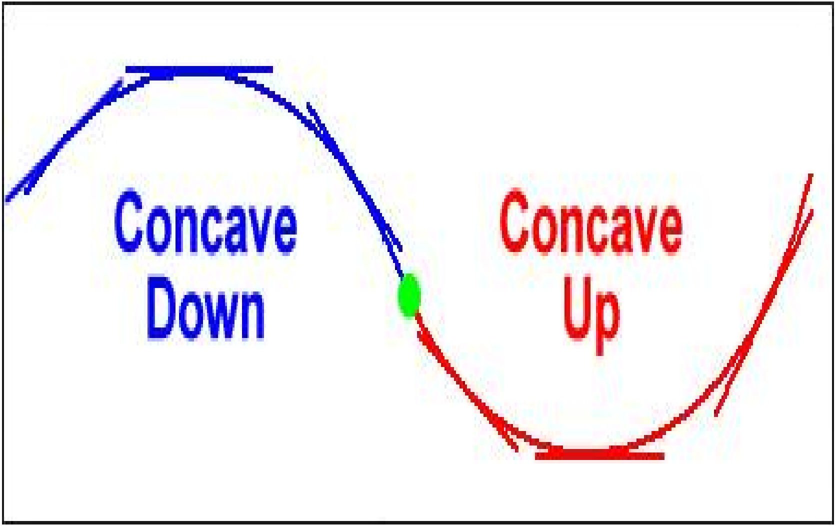


Figure: Plot of $f(x) = x^{1/3}(16 - x)$.

Concavity and The Second Derivative

Concavity: refers to the *bending* nature of a graph. In particular, a curve is concave down if it's cupped side is down, and it is concave up if it's cupped upward.

Concavity



Figure

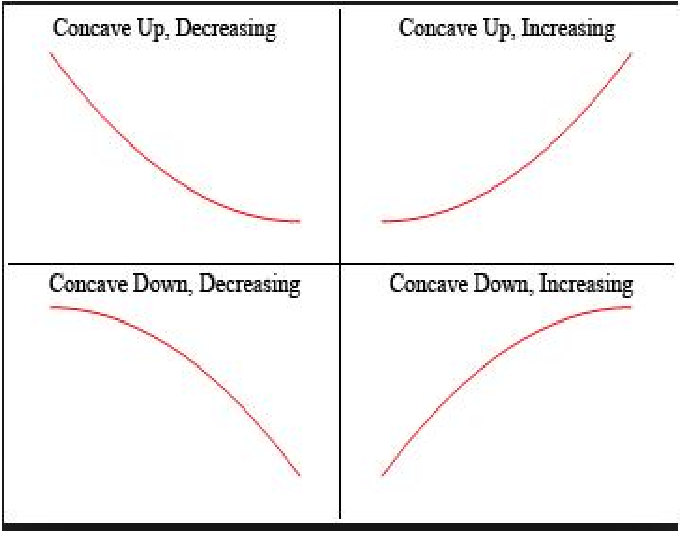


Figure: A graph can have either increasing or decreasing behavior and be either concave up or down.

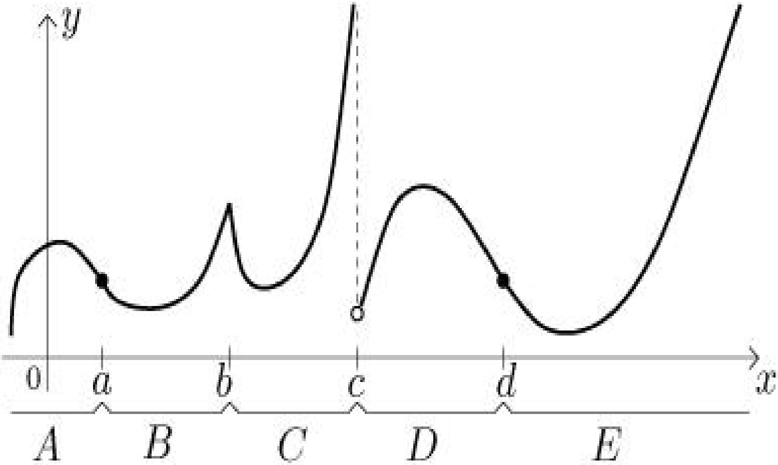


Figure: We can consider concavity at a point, but it's best thought of as a property over an interval. Many function's graphs have concavity that changes over the domain.

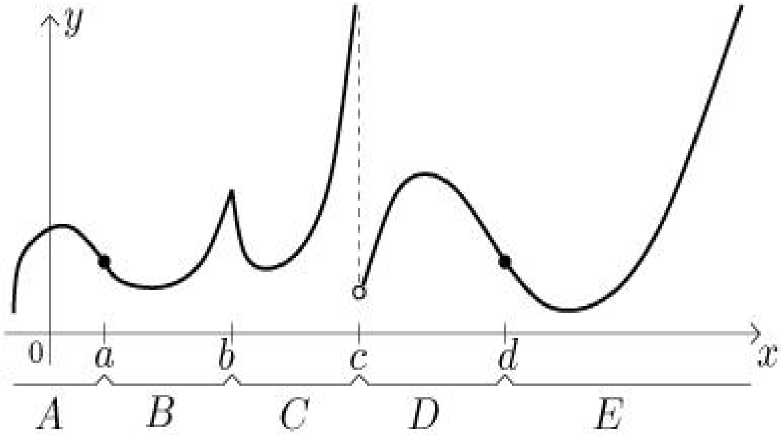
Definition of Concavity

If the graph of a function f lies above all of its tangent lines over an interval I, then f is concave up on I. If the graph of f lies below each of its tangent lines on an interval I, f is concave down on I.

Theorem: (Second Derivative Test for Concavity)
Suppose f is twice differentiable on an interval I.

- If $f''(x) > 0$ on I, then the graph of f is concave up on I.
- If $f''(x) < 0$ on I, then the graph of f is concave down on I.

Definition: A point P on a curve $y = f(x)$ is called an **inflection point** if f is continuous at P and the concavity of f changes at P (from down to up or from up to down). A point where $f''(x) = 0$ would be a candidate for being an inflection point.



Concavity and Extrema:

Theorem: (Second Derivative Test for Local Extrema)
Suppose $f'(c) = 0$ and that f'' is continuous near c. Then

- if $f''(c) > 0$, f takes a local minimum at c,
- if $f''(c) < 0$, then f takes a local maximum at c.

If $f''(c) = 0$, then the test fails. f may or may not have a local extrema. You can go back to the first derivative test to find out.

Example

Analyze the function $f(x) = \dfrac{x}{x^2+1}$ In particular, indicate

(a) the intervals on which f is increasing and decreasing,
(b) the intervals on which f is concave up and concave down,
(c) identify critical points and classify any local extrema, and
(d) identify any points of inflection.

$$f'(x) = \frac{(x^2+1) - x(2x)}{(x^2+1)^2} = \frac{x^2+1-2x^2}{(x^2+1)^2} \implies f'(x) = \frac{1-x^2}{(x^2+1)^2}$$

$$f''(x) = \frac{-2x(x^2+1)^2 - (1-x^2)\,2(x^2+1)(2x)}{(x^2+1)^4} = \frac{\cancel{(x^2+1)}\left[-2x(x^2+1) - 4x(1-x^2)\right]}{(x^2+1)^{\cancel{4}\,3}}$$

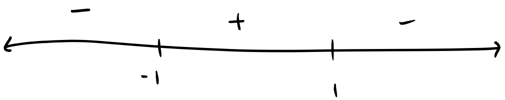
$$f''(x) = \frac{-2x^3 - 2x - 4x + 4x^3}{(x^2+1)^3} = \frac{2x^3 - 6x}{(x^2+1)^3} = \frac{2x(x^2-3)}{(x^2+1)^3}$$

$$f'(x) = \frac{1-x^2}{(x^2+1)^2} \qquad f''(x) = \frac{2x(x^2-3)}{(x^2+1)^3}$$

$f'(x) = 0$ if $1 - x^2 = 0$

$x = \pm 1$ (critical numbers)

Sign analysis on f'



$f'(-2)$ $\frac{-}{+}$ $\qquad f'(0)$ $\frac{+}{+}$ $\qquad f'(2)$ $\frac{-}{+}$

$f''(x) = 0 \Rightarrow 2x(x^2-3) = 0 \Rightarrow x = 0, \pm\sqrt{3}$

$$f''(x) = \frac{2x(x^2-3)}{(x^2+1)^3}$$

Sign analysis of f''

	$(-\infty, -\sqrt{3})$	$-\sqrt{3}$	$(-\sqrt{3}, 0)$	0	$(0, \sqrt{3})$	$\sqrt{3}$	$(\sqrt{3}, \infty)$
sign of f''	$-$		$+$		$-$		$+$

$f''(-2)\ \frac{(-)(+)}{+}$ $\qquad f''(-1)\ \frac{(-)(-)}{+}$ $\qquad f''(1)\ \frac{+(-)}{+}$ $\qquad f''(2)\ \frac{+(+)}{+}$

(a) f is increasing on $(-1, 1)$

f is decreasing on $(-\infty, -1) \cup (1, \infty)$

(b) f is concave up on $(-\sqrt{3}, 0) \cup (\sqrt{3}, \infty)$

f is concave down on $(-\infty, -\sqrt{3}) \cup (0, \sqrt{3})$

(c) Critical numbers -1 and 1

2nd Der. test $f''(-1) > 0$ f takes a local minimum @ -1

2nd Der. test $f''(1) < 0$

f takes a local maximum @ 1

(d) f has inflection points at

$-\sqrt{3}$, 0, and $\sqrt{3}$.

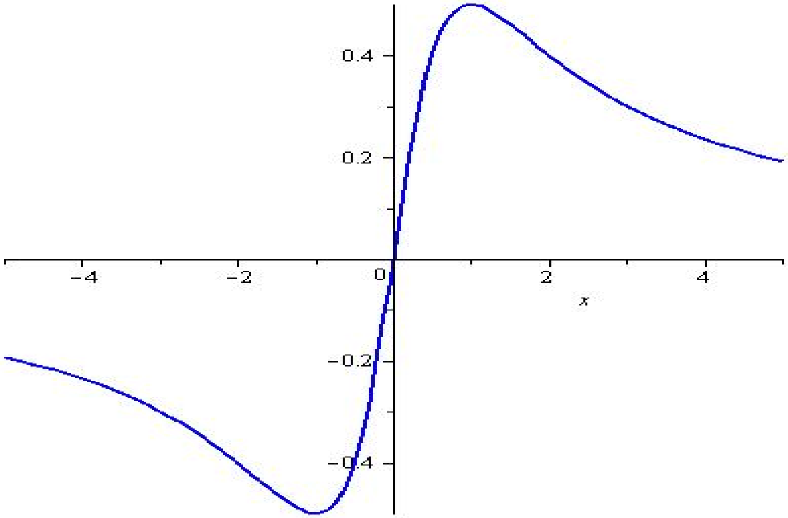


Figure: $y = \frac{x}{x^2+1}$