Oct 30 Math 2253H sec. 05H Fall 2014

Section 4.4: Indefinite Integrals and Net Change

New notation for antiderivatives:

If $F'(x) = f(x)$, i.e. F is any antiderivative of f, we will write

$$\int f(x)\, dx = F(x) + C$$

and we'll call $\int f(x)\, dx$ the **indefinite** integral of f.

Note:

$$\int_a^b f(x)\,dx$$

is called the "definite integral of f from a to b." And, it is a number.

$$\int f(x)\,dx$$

is called an "indefinite integral of f". And, it is a family of functions.

Table of Indefinite Integrals (things we already know)

$$\int cf(x)\,dx = c\int f(x)\,dx$$

$$\int (f(x) \pm g(x))\,dx = \int f(x)\,dx \pm \int g(x)\,dx$$

$$\int k\,dx = kx + C$$

$$\int x^n\,dx = \frac{x^{n+1}}{n+1} + C, \quad \text{for } n \neq -1$$

Table Continued...

$$\int \sin x \, dx = -\cos x + C, \qquad \int \cos x \, dx = \sin x + C$$

$$\int \sec^2 x \, dx = \tan x + C, \qquad \int \csc^2 x \, dx = -\cot x + C$$

$$\int \sec x \tan x \, dx = \sec x + C, \quad \int \csc x \cot x \, dx = -\csc x + C$$

Evaluate the definite integral

$$\int_{\pi/6}^{\pi/3} \cos^2\left(\frac{r}{2}\right)\, dr \qquad \text{Hint: } \cos^2\theta = \frac{1}{2} + \frac{1}{2}\cos(2\theta)$$

$$\cos^2\left(\frac{r}{2}\right) = \frac{1}{2} + \frac{1}{2}\cos\left(2\cdot\frac{r}{2}\right)$$

$$= \frac{1}{2} + \frac{1}{2}\cos(r)$$

$$= \int_{\pi/6}^{\pi/3} \left(\frac{1}{2} + \frac{1}{2}\cos(r)\right) dr$$

$$= \frac{1}{2}r + \frac{1}{2}\sin r \Big|_{\pi/6}^{\pi/3}$$

$$= \frac{1}{2}\cdot\frac{\pi}{3} + \frac{1}{2}\sin\left(\frac{\pi}{3}\right) - \left(\frac{1}{2}\cdot\frac{\pi}{6} + \frac{1}{2}\sin\left(\frac{\pi}{6}\right)\right)$$

$$= \frac{\pi}{6} + \frac{\sqrt{3}}{4} - \left(\frac{\pi}{12} + \frac{1}{4}\right) = \frac{\pi}{6} - \frac{\pi}{12} + \frac{\sqrt{3}}{4} - \frac{1}{4} = \frac{\pi}{12} + \frac{\sqrt{3}-1}{4}$$

We had

$$\int f(x)\,dx = F(x) + C \quad \text{means} \quad F'(x) = f(x).$$

And, according to the Fundamental Theorem of Calculus, if f is continuous on $[a, b]$ then

$$\frac{d}{dx}\int_a^x f(t)\,dt = f(x), \quad a \le x \le b, \quad \text{and}$$

$$\int_a^b f(x)\,dx = F(b) - F(a)$$

where F is any antiderivative of f.

A consequence is the ”Net Change” Theorem:

$$\int_a^b F'(x)\,dx = F(b) - F(a)$$

The integral of the rate of change is the net change of the function!

Example

A particle moves along the x-axis so that it's position $s(t)$ satisfies

$$s(0) = -3 \quad \text{and} \quad s(4) = 13.$$

Suppose the particle's velocity at time t is given by $v(t)$. Evaluate

$$\int_0^4 v(t)\,dt = \int_0^4 s'(t)\,dt = s(4) - s(0)$$

$$= 13 - (-3) = 16$$

Evaluate (looking ahead to section 4.5)

$$\int_0^1 2x(x^2+1)^2\,dx = \int_0^1 2x\,(x^4+2x^2+1)\,dx$$

$$= \int_0^1 (2x^5+4x^3+2x)\,dx$$

$$= 2\frac{x^6}{6} + 4\frac{x^4}{4} + 2\frac{x^2}{2}\Big|_0^1$$

$$= \frac{x^6}{3} + x^4 + x^2\Big|_0^1 = \frac{1^6}{3} + 1^4 + 1^2 - \left(\frac{0^6}{3} + 0^4 + 0^2\right)$$

$$= \frac{1}{3} + 1 + 1 = \frac{7}{3}$$

Suppose we wanted to evaluate

$$\int_0^1 2x(x^2+1)^{10}\,dx.$$

The same approach can be used. But it's really unpleasant!

Differentials (revisit of section 2.9)

Definition: Let f be a differentiable function of x. The variable

$$dx$$

is called a *differential*. It is an **independent** variable. Letting $y = f(x)$, the differential

$$dy$$

is a **dependent** variable defined by

$$dy = f'(x)dx.$$

In Liebniz notation

$$f'(x) = \frac{dy}{dx}$$

$$dy = \frac{dy}{dx}\,dx$$

Examples:

$dy = \frac{dy}{dx}\, dx$

(a) Given $y = \sin^2(x)$, express dy in terms of dx.

$\frac{dy}{dx} = 2\sin(x)\cos(x) \implies dy = 2\sin(x)\cos(x)\, dx$

(b) Given $u = x^2 + 2x$, express du in terms of dx.

$\frac{du}{dx} = 2x+2 \implies du = (2x+2)\, dx$

(c) Given $u = \frac{x}{3} + 1$, express du in terms of dx.

$$\frac{du}{dx} = \frac{1}{3} \qquad \Rightarrow \qquad du = \frac{1}{3}\, dx$$

(d) Given $v = \theta^8$, express dv in terms of $d\theta$.

$$\frac{dv}{d\theta} = 8\theta^7 \qquad \Rightarrow \qquad dv = 8\theta^7\, d\theta$$

$$\int_0^1 2x(x^2+1)^2\,dx = \frac{7}{3}$$

Evaluate this by letting $u = x^2 + 1$.

we'll write the integral in terms of u.

$u = x^2+1$ so

$du = 2x\,dx$

when $x=0$

$u = 0^2+1 = 1$

when $x=1$

$u = 1^2+1 = 2$

$$\int_0^1 (x^2+1)^2\, 2x\,dx = \int_1^2 (u)^2\,du$$

$$= \int_1^2 u^2\,du = \left.\frac{u^3}{3}\right|_1^2$$

$$= \frac{2^3}{3} - \frac{1^3}{3} = \frac{8}{3} - \frac{1}{3} = \frac{7}{3}$$

$$\int_0^1 2x(x^2+1)^{10}\, dx$$

Evaluate this by letting $u = x^2 + 1$.

$u = x^2 + 1$

$du = 2x\, dx$

$u = 1$ when $x = 0$

and

$u = 2$ when $x = 1$

$$\int_0^1 (x^2+1)^{10}\, 2x\, dx$$

$$= \int_1^2 (u)^{10}\, du$$

$$= \int_1^2 u^{10}\, du$$

$$= \frac{u^{11}}{11}\Big|_1^2 = \frac{2^{11}}{11} - \frac{1^{11}}{11}$$

$$= \frac{2048}{11} - \frac{1}{11} = \frac{2047}{11}$$

Section 4.5: The Substitution Rule

Theorem: Suppose $u = g(x)$ is a differentiable function, and f is continuous on the range of g. Then

$$\int f(g(x))\, g'(x)\, dx = \int f(u)\, du.$$

This is often refered to as ***u*-substitution.**
This is the Chain Rule in reverse!

For $u = g(x)$, $du = g'(x)\,dx$

and $f(g(x)) = f(u)$

Evaluate each Indefinite integral using Substitution as Needed

(a) $\displaystyle\int (3x+2)^3\, dx$

Let $u = 3x+2$

$du = 3\,dx$

$\frac{1}{3}\,du = dx$

$$= \int (u)^3 \frac{1}{3}\, du$$

$$= \frac{1}{3}\int u^3\, du = \frac{1}{3}\frac{u^4}{4} + C$$

$$= \frac{1}{12}u^4 + C$$

$$= \frac{1}{12}(3x+2)^4 + C$$