Oct 3 Math 2253H sec. 05H Fall 2014

Section 3.4 Limits at Infinity

We know what is meant by a limit being infinite (i.e. $f \to \infty$ or $f \to -\infty$). Now, we want to consider limits like

$$\lim_{x\to\infty} f(x) \qquad \text{or like}$$

$$\lim_{x\to-\infty} f(x).$$

What is meant by such a thing, and how is it related to a function's graph?

Definitions

Let f be defined on an interval (a, ∞). Then

$$\lim_{x\to\infty} f(x) = L$$

provided the value of f can be made arbitrarily close to L by taking x sufficiently large.

Similarly

Defintion: Let f be defined on an interval $(-\infty, a)$. Then

$$\lim_{x\to-\infty} f(x) = L$$

provided the value of f can be made arbitrarily close to L by taking x sufficiently large and negative.

A critical observation:

$$\lim_{x\to\infty} \frac{1}{x^r} = 0 \qquad \text{for any power } r > 0.$$

and

$$\lim_{x\to-\infty} \frac{1}{x^n} = 0 \qquad \text{for any integer } n = 1, 2, 3, \ldots$$

Evaluate the following limit.

$$\lim_{x\to\infty} \frac{3x^3+4x^2-2x+1}{2x-6x^2-2x^3}$$

• Identify the largest power of x in the denominator (n)

$$= \lim_{x\to\infty} \left(\frac{3x^3+4x^2-2x+1}{2x-6x^2-2x^3}\right)\cdot\frac{\frac{1}{x^3}}{\frac{1}{x^3}}$$

• multiply by

$$1 = \frac{\frac{1}{x^n}}{\frac{1}{x^n}}$$

$$= \lim_{x\to\infty} \frac{3+\frac{4}{x}-\frac{2}{x^2}+\frac{1}{x^3}}{\frac{2}{x^2}-\frac{6}{x}-2} = \frac{3+0-0+0}{0-0-2} = \frac{-3}{2}$$

Evaluate the following limit.

Same technique

$$\lim_{x\to-\infty} \frac{3x^3+4x^2-2x+1}{2x-6x^2-2x^3}$$

$$= \lim_{x\to-\infty} \left(\frac{3x^3+4x^2-2x+1}{2x-6x^2-2x^3}\right)\cdot\frac{\frac{1}{x^3}}{\frac{1}{x^3}}$$

$$= \lim_{x\to-\infty} \frac{3+\frac{4}{x}-\frac{2}{x^2}+\frac{1}{x^3}}{\frac{2}{x^2}-\frac{6}{x}-2} = \frac{3}{-2} = \frac{-3}{2}$$

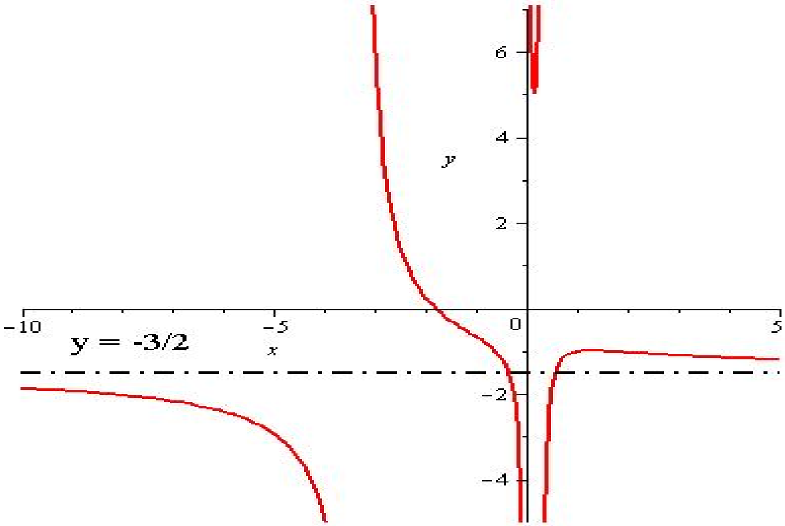

Figure: plot of $f(x) = \frac{3x^3+4x^2-2x+1}{2x-6x^2-2x^3}$ along with the line $y = -3/2$

Definition (Horizontal Asymptote)

The line $y = L$ is a horizontal asymptote to the graph of f if

$$\lim_{x\to\infty} f(x) = L \quad \text{or} \quad \lim_{x\to-\infty} f(x) = L.$$

If f is a rational function, and one of these limits is a finite number L, then both are this same finite number. Other types of functions may have different limits (even finite ones) at the two extremes. This means that there are functions with two different horizontal asymptotes!

Evaluate the limit.

(a) $$\lim_{x\to\infty} \frac{\sqrt{x^2+1}}{x-2}$$

$$= \lim_{x\to\infty} \left(\frac{\sqrt{x^2+1}}{x-2}\right)\frac{\frac{1}{x}}{\frac{1}{x}}$$

$$= \lim_{x\to\infty} \frac{\sqrt{x^2+1}\left(\frac{1}{\sqrt{x^2}}\right)}{1-\frac{2}{x}}$$

$$= \lim_{x\to\infty} \frac{\sqrt{(x^2+1)\frac{1}{x^2}}}{1-\frac{2}{x}}$$

For $x>0$ then

$$x=\sqrt{x^2}$$

i.e. $\frac{1}{x}=\frac{1}{\sqrt{x^2}}$

for $x>0$

$$\sqrt{a}\cdot\frac{1}{\sqrt{b}}=\sqrt{\frac{a}{b}}$$

$$= \lim_{x \to \infty} \frac{\sqrt{1 + \frac{1}{x^2}}}{1 - \frac{2}{x}} = \frac{\sqrt{1+0}}{1-0} = 1$$

(b) $\lim_{x\to-\infty} \frac{\sqrt{x^2+1}}{x-2}$

$$= \lim_{x\to-\infty} \left(\frac{\sqrt{x^2+1}}{x-2}\right)\cdot\frac{\frac{1}{x}}{\frac{1}{x}}$$

$$= \lim_{x\to-\infty} \frac{\sqrt{x^2+1}\left(\frac{-1}{\sqrt{x^2}}\right)}{1-\frac{2}{x}}$$

$$= \lim_{x\to-\infty} \frac{-\sqrt{(x^2+1)\frac{1}{x^2}}}{1-\frac{2}{x}}$$

For $x<0$ then

$$x=-\sqrt{x^2}$$

i.e.

$$\frac{1}{x}=\frac{-1}{\sqrt{x^2}} \quad \text{for } x<0$$

$$= \lim_{x \to -\infty} \frac{-\sqrt{1 + \frac{1}{x^2}}}{1 - \frac{2}{x}} = \frac{-\sqrt{1+0}}{1-0} = -1$$

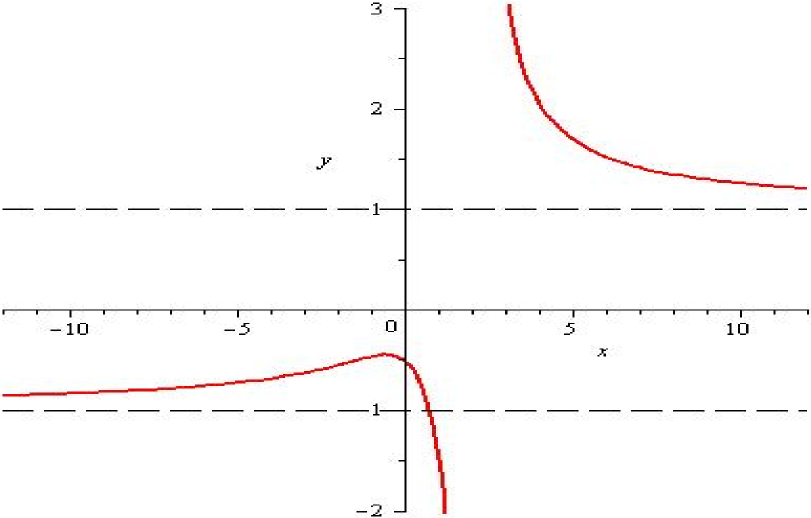

Figure: plot of $f(x) = \frac{\sqrt{x^2+1}}{x-2}$ along with the two lines $y = 1$ and $y = -1$.

Vertical and Horizontal Asymptotes

Recall: The line $x = c$ is a *vertical asymptote* to the graph of f if

$$\lim_{x \to c^+} f(x) = \pm\infty, \quad \text{or} \quad \lim_{x \to c^-} f(x) = \pm\infty.$$

The line $y = L$ is a *horizontal asymptote* to the graph of f if

$$\lim_{x \to \infty} f(x) = L, \quad \text{or} \quad \lim_{x \to -\infty} f(x) = L.$$

A good candidate for a vertical asymptote would be a number that makes a denominator zero.

Find any vertical and horizontal asymptotes to the graph of

$$f(x) = \frac{2x^2 - 4x - 6}{x^2 - 3x - 4} = \frac{2(x+1)(x-3)}{(x+1)(x-4)}$$

Candidates for vertical asymptotes are -1 and 4.

$$\lim_{x\to -1} f(x) = \lim_{x\to -1} \frac{2(x+1)(x-3)}{(x+1)(x-4)} = \lim_{x\to -1} \frac{2(x-3)}{x-4} = \frac{8}{5}$$

There is not a vertical asymptote @ $x=-1$

$$\lim_{x\to 4^-} f(x) = \lim_{x\to 4^-} \frac{2(x+1)(x-3)}{(x+1)(x-4)} = -\infty$$

The line $x=4$ is a vertical asymptote.

Horizontal:

$$\lim_{x\to\infty} f(x) = \lim_{x\to\infty} \frac{2x^2-4x-6}{x^2-3x-4}$$

$$= \lim_{x\to\infty} \left(\frac{2x^2-4x-6}{x^2-3x-4}\right) \cdot \frac{\frac{1}{x^2}}{\frac{1}{x^2}}$$

$$= \lim_{x\to\infty} \frac{2 - \frac{4}{x} - \frac{6}{x^2}}{1 - \frac{3}{x} - \frac{4}{x^2}} = \frac{2}{1} = 2$$

Since f is rational, $\lim_{x\to-\infty} f(x) = 2$ as well.

f has one vertical asymptote $x=4$

and one horizontal asymptote $y=2$

to its graph.