Oct 6 Math 2253H sec. 05H Fall 2014

Section 3.4 Limits at Infinity

Definition: Let f be defined on an interval (a, ∞). Then

$$\lim_{x\to\infty} f(x) = L$$

provided the value of f can be made arbitrarily close to L by taking x sufficiently large.

Similarly

Defintion: Let f be defined on an interval $(-\infty, a)$. Then

$$\lim_{x\to-\infty} f(x) = L$$

provided the value of f can be made arbitrarily close to L by taking x sufficiently large and negative.

Formal Definition

Definition: Let f be defined on an interval (a, ∞). Then

$$\lim_{x\to\infty} f(x) = L$$

provided for every number $\epsilon > 0$ there exists a number $M > 0$ such that

$$\text{if} \quad x > M, \quad \text{then} \quad |f(x) - L| < \epsilon.$$

Similarly, for f defined on $(-\infty, a)$

$$\lim_{x\to-\infty} f(x) = L$$

provided for every number $\epsilon > 0$ there exists a number $M > 0$ such that

$$\text{if} \quad x < -M, \quad \text{then} \quad |f(x) - L| < \epsilon.$$

Proof that $\lim_{x\to\infty} \frac{1}{x} = 0$.

$f(x) = \frac{1}{x}$ $L = 0$

Need to find $M > 0$ such that

$\left|\frac{1}{x} - 0\right| < \varepsilon$ whenever $x > M$

for any $\varepsilon > 0$.

Note $\left|\frac{1}{x} - 0\right| = \left|\frac{1}{x}\right|$ so $\frac{1}{|x|} < \varepsilon$ requires

(divide by ε, mult by $|x|$)

$\frac{1}{\varepsilon} < |x|$

for $x>0$ we need $x > \frac{1}{\varepsilon}$

So we can take $M = \frac{1}{\varepsilon}$.

Proof: Let $\varepsilon > 0$. Set $M = \frac{1}{\varepsilon}$ so that $M > 0$.

Note that if $x > M$, then

$$x > \frac{1}{\varepsilon} \implies \varepsilon > \frac{1}{x} = \frac{1}{|x|} \quad (\text{since } x>0)$$

That is $\left|\frac{1}{x} - 0\right| < \varepsilon$.

Hence $\lim_{x\to\infty} \frac{1}{x} = 0$.

This argument is easily modified to show that

$$\lim_{x\to\infty} \frac{1}{x^r} = 0 \quad \text{for any } r>0.$$

Infinite limits at infinity

We may have a limit such as

$$\lim_{x\to\infty} f(x) = \infty, \quad \text{or} \quad \lim_{x\to-\infty} f(x) = \infty.$$

The right hand sides may also be $+\infty$ or $-\infty$.

This means that $f(x)$ becomes unboundedly large (positively or negatively) as x becomes unboundedly large.

Examples

$$\lim_{x\to\infty} x^2 = \infty$$

$$\lim_{x\to-\infty} x^3 = -\infty$$

$$\lim_{x\to\infty}(x^2-x) = \lim_{x\to\infty} x(x-1) = \infty$$

"$\infty - \infty$" is not a number

$$\lim_{x\to\infty}(x^4-x^6) = \lim_{x\to\infty} x^4(1-x^2) = -\infty$$

The indeterminate form $\infty - \infty$

Evaluate

$$\lim_{x\to\infty}(x-\sqrt{x^2+3x})$$

$$= \lim_{x\to\infty} \left(x-\sqrt{x^2+3x}\right)\cdot\frac{\left(x+\sqrt{x^2+3x}\right)}{x+\sqrt{x^2+3x}}$$

$$= \lim_{x\to\infty} \frac{x^2-(x^2+3x)}{x+\sqrt{x^2+3x}}$$

$$= \lim_{x\to\infty} \frac{-3x}{x+\sqrt{x^2+3x}}$$

We can use

$$\lim_{x\to\infty}\frac{1}{x^r}=0 \quad \text{for } r>0$$

If we get rid of the radical

Use the conjugate

$$= \lim_{x\to\infty} \frac{-3x}{x+\sqrt{x^2+3x}} \cdot \frac{\frac{1}{x}}{\frac{1}{x}}$$

for $x > 0$

$x = \sqrt{x^2}$

$\frac{1}{x} = \frac{1}{\sqrt{x^2}}$

$$= \lim_{x\to\infty} \frac{-3}{1+\sqrt{(x^2+3x)\frac{1}{x^2}}}$$

$$= \lim_{x\to\infty} \frac{-3}{1+\sqrt{1+\frac{3}{x}}} = \frac{-3}{1+\sqrt{1+0}} = \frac{-3}{2}$$

Section 3.5: Graphing

Analyze the function $f(x) = 5x^{2/3} - 2x^{5/3}$, and use the results to produce a rough plot of the graph $y = f(x)$.

$$f(x) = 5x^{2/3} - 2x^{5/3}$$

the domain is all reals

There are no vertical asymptotes.

Look for horizontal asymptotes:

$$\lim_{x\to\infty} f(x) = \lim_{x\to\infty} 5x^{2/3} - 2x^{5/3} = \lim_{x\to\infty} x^{2/3}(5-2x) = -\infty$$

$$\lim_{x\to-\infty} f(x) = \lim_{x\to-\infty} x^{2/3}(5-2x) = \infty$$

$\boxed{\text{f has no horizontal asymptotes but it goes down at the right and up at the far left.}}$

<u>Intercepts:</u> $\boxed{\text{y-intercept } (0,0)}$ $f(0) = 5(0)^{2/3} - 2(0)^{5/3} = 0$

x-intercept(s) $f(x) = 0 \Rightarrow x^{2/3}(5-2x) = 0$

There are 2 of them $(0,0)$ and $\left(\frac{5}{2}, 0\right)$.