Oct 9 Math 2253H sec. 05H Fall 2014

Section 3.7: Applied Optimization

Example: A 216 m^2 rectangular pea patch is to be enclosed by a fence and divided into two equal parts by another fence parallel to one of its sides. What dimensions of the outer rectangle will require the smallest total length of fencing and how much fencing will be needed?

Our Work on this Example:

We let x and y be the length and width of a rectangular patch, respectively (each in meters). The area constraint gives

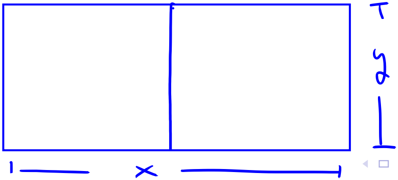
$$xy = 216 \text{ m}^2 \quad \Longrightarrow \quad x = \frac{216}{y}.$$

The total amount of fencing needed was

$$F = 2x + 3y \text{ m}.$$

So given the constraint we want to minimize the objective function

$$F(y) = \frac{432}{y} + 3y \quad y > 0.$$



Find minimum F:

Crit #: $F'(y) = -432y^{-2} + 3 = \dfrac{-432}{y^2} + 3$

$F'(y)$ DNE if $y=0$ not of interest since $y>0$

$F'(y) = 0 \Rightarrow \dfrac{-432}{y^2} + 3 = 0 \Rightarrow 3 = \dfrac{432}{y^2}$

$\Rightarrow y^2 = \dfrac{432}{3} = 144 \Rightarrow y = 12$ or $y = -12$

One critical number for $y>0$ is $y=12$.

Verify that $F(12)$ is a minimum:

2^{nd} Der. test:

$$F''(y) = -2(-432)y^{-3} = \frac{864}{y^3}$$

$$F''(12) = \frac{864}{12^3} > 0$$ F is concave up

F(12) is a minimum!

The minimum for y is 12.

For x it is $\frac{216}{12} = 18$

When $x=18$ and $y=12$

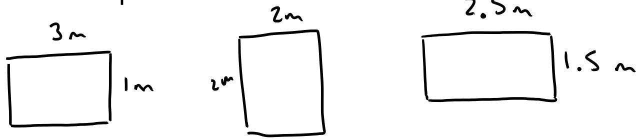
$$F = 2(18) + 3(12) \text{ m} = (36 + 36) \text{ m} = 72 \text{ m}$$

The dimensions are $12\text{m} \times 18\text{ m}$ with

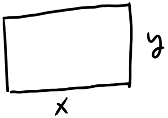
72 m of fencing needed.

Applied Optimization Example

Show that among all rectangles with perimeter 8m, the one with the largest area is a square.



Generic rectangle



Let x, y be the length and width in m.

Area $A = xy$

we wish to maximize A

objective function

Perimeter $P = 2x + 2y$

we require $P = 8$ i.e.

$$2x + 2y = 8$$

constraint

From the constraint: $2y = 8 - 2x \Rightarrow y = 4 - x$

As a function of x alone

$$A(x) = x(4-x) = 4x - x^2 \qquad 0 < x < 4$$

Find Crit # $\quad A'(x) = 4 - 2x \qquad A'(x)$ DNE never

$$A'(x) = 0 \Rightarrow 4 - 2x = 0 \Rightarrow x = 2$$

2nd Der. test : $A''(x) = -2$ so $A''(2) = -2 < 0$

A takes a local max at $x = 2$

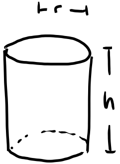
The optimal length is $2m$.

The optimal width is $y = (4-2)m = 2m$

So the optimum is a square.

Applied Optimization Example

A can in the shape of a right circular cylinder is to have a volume of 128π cubic cm. The material that the top and bottom are made of costs \$0.20/cm^2 and the material that the lateral surface is made of costs \$0.10/cm^2. Find the dimensions of the can that minimize the total cost of production.



Let r and h be the base radius and height of such a cylinder (in cm).

Volume $V = \pi r^2 h$ given $V = 128\pi$ cm^3

Build cost function:

Top + Bottom

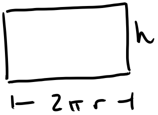
πr^2 each

Cost = area × Cost/unit area

$$\text{Cost} = 2\pi r^2 \text{ cm}^2 \times \left(20 \frac{¢}{\text{cm}^2}\right)$$

$$= 40\pi r^2 \ ¢$$

Lateral surface



$$\text{Cost} = 2\pi r h \ \text{cm}^2 \cdot \left(10 \ \frac{¢}{\text{cm}^2}\right)$$

$$= 20\pi r h \ ¢$$

Cost for one can $\quad C = 40\pi r^2 + 20\pi r h$ objective

From $V = 128\pi \qquad \pi r^2 h = 128\pi \implies$

$$h = \frac{128}{r^2}$$

As a function of only $r \qquad C(r) = 40\pi r^2 + 20\pi r\left(\frac{128}{r^2}\right)$

$$C(r) = 40\pi r^2 + \frac{2560\pi}{r}, \ r > 0$$ minimize this

Find crit # $\quad C'(r) = 80\pi r - \frac{2560\pi}{r^2}$

$C'(r)$ DNE if $r = 0$ – not of interest here

$$C'(r) = 0 \implies 80\pi r - \frac{2560\pi}{r^2} = 0 \implies$$

$$80\pi r = \frac{2560\pi}{r^2}$$

$$r^3 = \frac{2560\pi}{80\pi} = 32$$

$$r = \sqrt[3]{32} = 2\sqrt[3]{4}$$

Verify we have a minimum

$$C''(r) = 80\pi + \frac{2(2560)\pi}{r^3}$$

$$C''(\sqrt[3]{32}) = 80\pi + \frac{2(2560\pi)}{32} > 0$$

$C(\sqrt[3]{32})$ is a minimum.

$$h = \frac{128}{\left(\sqrt[3]{32}\right)^2} = \frac{128}{32^{2/3}}$$

The dimensions for minimum cost are

$$r = \sqrt[3]{32} \text{ cm} \qquad h = \frac{128}{32^{2/3}} \text{ cm}$$

$$\approx 3.17 \text{ cm} \qquad \approx 12.7 \text{ cm}$$