Sept 11 Math 2253H sec. 05H Fall 2014

Section 2.7: Applications in Various Sciences

Business Suppose a manufacturer can produce x units of some commodity at a total cost $C(x)$. If production is changed from a quantity of x_1 to a quantity of x_2, then the average rate of change of cost is

$$\frac{\Delta C}{\Delta x} = \frac{C(x_2) - C(x_1)}{x_2 - x_1}.$$

The quantity referred to as *marginal cost* by economists is

$$\text{marginal cost} = \frac{dC}{dx}.$$

For profit P and revenue R, similarly $\frac{dP}{dx}$ and $\frac{dR}{dx}$ are called *marginal profit* and *marginal revenue*, respectively.

Example:

To produce a certain widget, it costs \$5000 just to access the production machinery . It then costs \$ $x(2 - 0.1x + 0.001x^2)$ in materials costs to produce x widgets. Determine the cost function $C(x)$ of producing x widgets.

$$C(x) = 5000 + x(2 - 0.1x + 0.001x^2) = 0.001x^3 - 0.1x^2 + 2x + 5000$$

Find the marginal cost of production $C'(100)$ when producing 100 widgets. What does this number predict?

$$C'(x) = 0.001(3x^2) - 0.1(2x) + 2$$

$$C'(x) = 3(0.001)x^2 - 2(0.1)x + 2$$

So

$$C'(100) = 3(0.001)(100)^2 - 2(0.1)(100) + 2$$

$$= 30 - 20 + 2 = 12$$

The marginal cost @ 100 widget production is

$$\left.\frac{dC}{dx}\right|_{100} = 12 \ \frac{\$}{\text{widget}}$$

The cost of producing one extra unit @ the 100 production level is expected to be about \$12.

Section 2.6: Implicit differentiation

The chain rule states that for a differentiable composition $f(g(x))$

$$\frac{d}{dx}\, f(g(x)) = f'(g(x))g'(x)$$

For $y = f(u)$ and $u = g(x)$

$$\frac{dy}{dx} = \frac{dy}{du}\,\frac{du}{dx}$$

Example

Assume f is a differentiable function of x. Find an expression for the derivative:

$$\frac{d}{dx}\left(f(x)\right)^2 = 2\left(f(x)\right)^1 f'(x)$$

$$= 2f(x)f'(x)$$

$$\frac{d}{dx}\tan\left(f(x)\right) = \sec^2\left(f(x)\right)f'(x)$$

Example

Suppose we know that $y = f(x)$ for some differentiable function (but we don't know exactly what f is). Find an expression for the derivative.

$$\frac{d}{dx}\sqrt{y} = \frac{d}{dx} y^{1/2} = \frac{1}{2} y^{-1/2} \frac{dy}{dx} = \frac{1}{2\sqrt{y}} \frac{dy}{dx}$$

$$\frac{d}{dx} x^2y^2 = 2x y^2 + x^2 (2y) \frac{dy}{dx}$$

$$= 2xy^2 + 2x^2y \frac{dy}{dx}$$

Implicitly defined functions

A relation—an equation involving two variables x and y—such as

$$x^2 + y^2 = 16 \quad \text{or} \quad (x^2 + y^2)^3 = x^2$$

implies that y is defined to be one or more functions of x.

$$x^2+y^2=16 \implies y^2 = 16-x^2 \implies y=\sqrt{16-x^2} \quad \text{OR}$$

$$y=-\sqrt{16-x^2}$$

two functions are implied

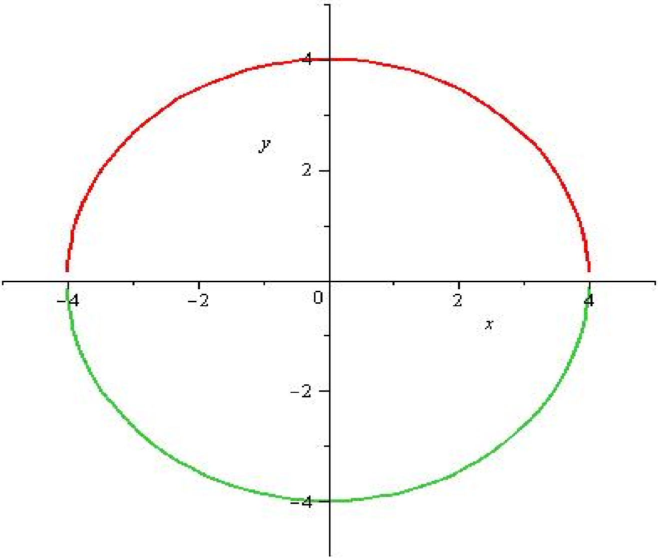

Figure: $x^2 + y^2 = 16$

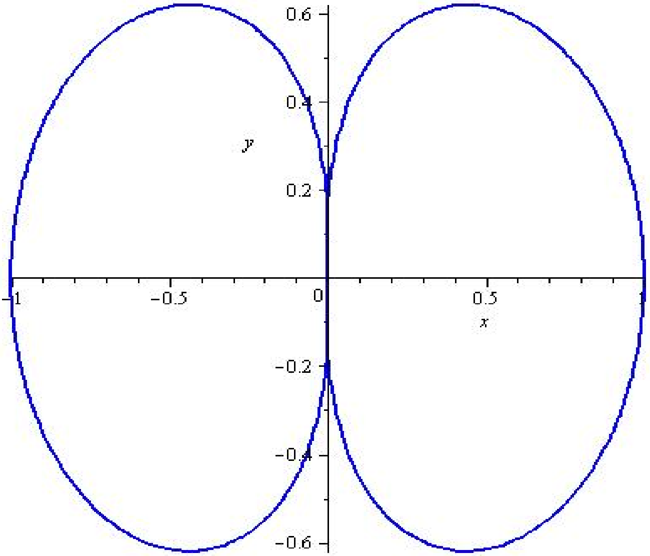

Figure: $(x^2 + y^2)^3 = x^2$

Explicit -vs- Implicit

A function is defined **explicitly** when given in the form

$$y = f(x).$$

e.g. $y = x^2$ or $y = 3x\sec^2(2x)$

A function is defined *implicitly* when it is given as a relation

$$F(x, y) = C,$$

for constant C.

e.g. $(x^2+y^2)^3 - x^2 = 0$ OR $y\tan(xy) + x^2 = 1$

Implicit Differentiation

Since $x^2 + y^2 = 16$ *implies* that y is a function of x, we can consider it's derivative.

$$\text{Find } \frac{dy}{dx} \text{ given } \quad x^2 + y^2 = 16.$$

Take $\frac{d}{dx}$ of both sides: $\quad \frac{d}{dx}(x^2+y^2) = \frac{d}{dx}(16)$

$$2x + 2y\frac{dy}{dx} = 0$$ isolate $\frac{dy}{dx}$

$$2y\frac{dy}{dx} = -2x \implies \frac{dy}{dx} = \frac{-2x}{2y} \implies \boxed{\frac{dy}{dx} = \frac{-x}{y}}$$

Show that the same result is obtained knowing

$$y = \sqrt{16 - x^2} \quad \text{or} \quad y = -\sqrt{16 - x^2}.$$

$y = \sqrt{16 - x^2}$ take $\frac{d}{dx}$ of both sides

$= (16 - x^2)^{1/2}$

$$\frac{dy}{dx} = \frac{1}{2}(16 - x^2)^{-1/2}(-2x)$$

$$\frac{dy}{dx} = \frac{-x}{\sqrt{16 - x^2}} = \frac{-x}{y}$$

$$y = -\sqrt{16-x^2} = -(16-x^2)^{1/2}$$

$$\frac{dy}{dx} = -\frac{1}{2}(16-x^2)^{-1/2}(-2x)$$

$$= \frac{x}{\sqrt{16-x^2}} = \frac{x}{-y}$$

$$\frac{dy}{dx} = \frac{-x}{y} \quad \text{again!}$$

Example

$$\text{Find } \frac{dy}{dx} \text{ given } \quad x^2 - 3xy + y^2 = y.$$

Take $\frac{d}{dx}$ of both sides $\quad \frac{d}{dx}(x^2 - 3xy + y^2) = \frac{d}{dx} y$

$$2x - 3\left(1y + x\frac{dy}{dx}\right) + 2y\frac{dy}{dx} = \frac{dy}{dx}$$

$$2x - 3y - 3x\frac{dy}{dx} + 2y\frac{dy}{dx} = \frac{dy}{dx}$$

$$-3x\frac{dy}{dx} + 2y\frac{dy}{dx} - \frac{dy}{dx} = 3y - 2x$$

$$(-3x + 2y - 1)\frac{dy}{dx} = 3y - 2x \implies \boxed{\frac{dy}{dx} = \frac{3y - 2x}{-3x + 2y - 1}}$$

Finding a Derivative Using Implicit Differentiation:

- Take the derivative of both sides of an equation with respect to the independent variable.
- Use all necessary rules for differenting powers, products, quotients, trig functions, compositions, etc.
- Remember the chain rule for each term involving the dependent variable (e.g. mult. by $\frac{dy}{dx}$ as required).
- Use necessary algebra to isolate the desired derivative.

Example

Find $\frac{dy}{dx}$.

$$\sin(x+y) = 2x$$

$$\cos(x+y)\,\frac{d}{dx}(x+y) = 2$$

$$\cos(x+y)\left(1+\frac{dy}{dx}\right) = 2 \implies$$

$$1+\frac{dy}{dx} = \frac{2}{\cos(x+y)} \implies$$

$$\boxed{\frac{dy}{dx} = 2\sec(x+y) - 1}$$