Sept 15 Math 2253H sec. 05H Fall 2014

Recap of the Chain Rule

Suppose t is an independent variable. If u is a function of t, and y is a function of u, then y is in turn a function of t.

When these functions are differentiable

$$\frac{dy}{dt} = \frac{dy}{du}\frac{du}{dt}.$$

Section 2.8 Related Rates

Motivating Example: A spherical balloon is being filled with air. Suppose that we know that the radius is increasing in time at a constant rate of 2 mm/sec. Can we determine the rate at which the surface area of the balloon is increasing at the moment that the radius is 10 cm?

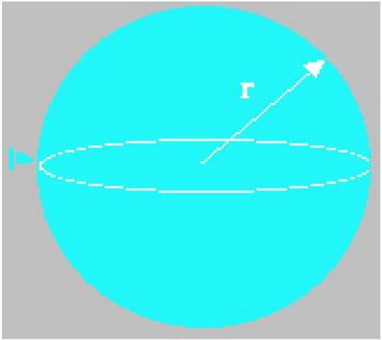


Figure: Spherical Balloon

Example Continued...

Suppose that the radius r and surface area $S = 4\pi r^2$ of a sphere are differentiable functions of time. Write an equation that relates

$$\frac{dS}{dt} \quad \text{to} \quad \frac{dr}{dt}.$$

By the chain rule $\quad \frac{dS}{dt} = \frac{dS}{dr}\frac{dr}{dt}$

$$S = 4\pi r^2 \implies \frac{dS}{dr} = 4\pi(2r) = 8\pi r$$

So $\quad \frac{dS}{dt} = 8\pi r \frac{dr}{dt}$

Given this result, find the rate at which the surface area is changing when the radius is 10 cm.

$$\frac{dS}{dt} = 8\pi r \frac{dr}{dt}$$

Givens: $r = 10$ cm (at moment of interest)

$$\frac{dr}{dt} = 2 \frac{\text{mm}}{\text{sec}}$$

$r = 10$ cm $= 100$ mm

$\frac{dS}{dt}$ is the rate of change of surface area

When $r = 100$ mm

$$\frac{dS}{dt} = 8\pi (100 \text{ mm})\, 2 \frac{\text{mm}}{\text{sec}}$$

$$= 1600\pi \frac{\text{mm}^2}{\text{sec}}$$

Surface area is changing @ 1600π sq. mm per second.

Example

A right circular cone of height h and base radius r has volume

$$V = \frac{\pi}{3} r^2 h.$$

(a) Find $\frac{dV}{dt}$ in terms of $\frac{dh}{dt}$ if r is constant.

Take $\frac{d}{dt}$ of both sides

$$\frac{dV}{dt} = \frac{\pi}{3} r^2 \frac{dh}{dt}$$

Example Continued...

$$V = \frac{\pi}{3} r^2 h$$

(b) Find $\frac{dV}{dt}$ in terms of $\frac{dr}{dt}$ if h is constant.

Take $\frac{d}{dt}$ of both sides:

$$\frac{dV}{dt} = \frac{\pi}{3} h (2r) \frac{dr}{dt} = \frac{2\pi}{3} rh \frac{dr}{dt}$$

And Continued Further...

$$V = \frac{\pi}{3} r^2 h$$

(c) Find $\frac{dV}{dt}$ in terms of $\frac{dh}{dt}$ and $\frac{dr}{dt}$ assuming neither r nor h is constant.

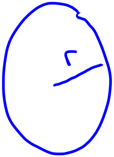
$$\frac{dV}{dt} = \frac{\pi}{3} r^2 \frac{dh}{dt} + \frac{\pi}{3} h (2r) \frac{dr}{dt}$$

$$\frac{dV}{dt} = \frac{\pi}{3} r^2 \frac{dh}{dt} + \frac{2\pi}{3} r h \frac{dr}{dt}$$

Example

When a circular plate is heated, its radius increases at a rate of 0.01 cm/min. Find the rate at which the area is increasing when the radius is 50 cm.

Let the radius and area be denoted by r and A, respectively.



Rates: $\frac{dA}{dt}$, $\frac{dr}{dt}$

Given $\frac{dr}{dt} = 0.01 \frac{cm}{min}$

Q: $\frac{dA}{dt} = ?$ when $r = 50\,cm$

From geometry, $A = \pi r^2$.

So $$\frac{dA}{dt} = \pi(2r)\frac{dr}{dt} = 2\pi r\frac{dr}{dt}$$

When $r = 50$ cm,

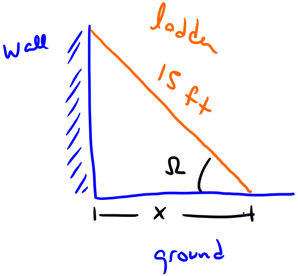
$$\frac{dA}{dt} = 2\pi(50\text{ cm})\left(0.01\ \frac{\text{cm}}{\text{min}}\right) = \frac{100\pi}{100}\ \frac{\text{cm}^2}{\text{min}}$$

$$\frac{dA}{dt} = \pi\ \frac{\text{cm}^2}{\text{min}}$$

A is increasing at a rate of π sq. cm per minute.

Example

A 15 ft ladder is sliding down a wall. When the base of the ladder is 12 ft from the wall, it is sliding at a rate of 3 ft/sec. At what rate is the angle between the base of the ladder and the ground changing when the base is 12 feet from the wall?



Let x be the distance between the base of the ladder and the wall. Let Ω be the angle between the ladder and ground (as shown).

rates: $\frac{dx}{dt}$, $\frac{d\Omega}{dt}$

Given: $\frac{dx}{dt} = 3\ \frac{ft}{sec}$ when $x = 12$ ft

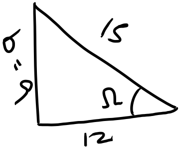
Q: $\frac{d\Omega}{dt} = ?$ when $x = 12$ ft

From trig $\cos\Omega = \frac{x \text{ ft}}{15 \text{ ft}} = \frac{1}{15}x$

Take $\frac{d}{dt}$ of both sides

$$(-\sin\Omega)\frac{d\Omega}{dt} = \frac{1}{15}\frac{dx}{dt}$$

Need $\sin\Omega$ when $x = 12$



$$b^2 = 15^2 - 12^2 = 225 - 144 = 81$$

$\sin\Omega = \frac{9}{15} = \frac{3}{5}$ (opp / hyp)

$$\frac{d\Omega}{dt} = -\frac{1}{15\sin\Omega}\frac{dx}{dt}$$

when $x = 12\,ft$

$$\frac{d\Omega}{dt} = \frac{-1}{15\,ft\left(\frac{3}{5}\right)}\,3\,\frac{ft}{sec} = \frac{-1}{3}\,\frac{1}{sec}$$

So Ω is decreasing at a rate of $\frac{1}{3}$ radians per second.