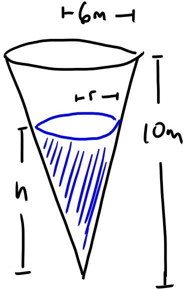
Example

A reservoir in the shape of an inverted right circular cone has height 10m and base radius 6m. If water is flowing into the reservoir at a constant rate of 50m^3/min. What is the rate at which the height of the water is increasing when the height is 5m?



Let r and h be the radius and height of the water, respectively.

If the volume of wate is V, then

$$V = \frac{\pi}{3} r^2 h$$

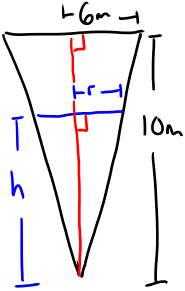
Rates: $\frac{dr}{dt}$, $\frac{dh}{dt}$

$\frac{dV}{dt}$

Given: $\frac{dV}{dt} = 50 \ \frac{m^3}{min}$

Q: $\frac{dh}{dt} = ?$ when $h = 5\,m$

Cross section



By similar triangles

$$\frac{r}{h} = \frac{6}{10} \Rightarrow r = \frac{3}{5}h$$

So $V = \frac{\pi}{3}\left(\frac{3}{5}h\right)^2 h \Rightarrow V = \frac{3\pi}{25}h^3$

Take $\frac{d}{dt}$ of both sides: $\frac{dV}{dt} = \frac{3\pi}{25}(3h^2)\frac{dh}{dt}$

$$\frac{dV}{dt} = \frac{9\pi}{25} h^2 \frac{dh}{dt} \implies \frac{dh}{dt} = \frac{dV}{dt}\left(\frac{25}{9\pi}\right)\frac{1}{h^2}$$

When $h = 5m$

$$\frac{dh}{dt} = \left(50\,\frac{m^3}{min}\right)\left(\frac{25}{9\pi}\right)\frac{1}{(5m)^2}$$

$$= 50\,\frac{m^{\cancel{3}}}{min}\cdot\frac{\cancel{25}}{9\pi}\cdot\frac{1}{\cancel{25}\,\cancel{m^2}}$$

$$= \frac{50}{9\pi}\,\frac{m}{min} \approx 1.77\,\frac{m}{min}$$

The height is increasing at a rate of $\frac{50}{9\pi}\,\frac{m}{min}$ when the height is $5m$.

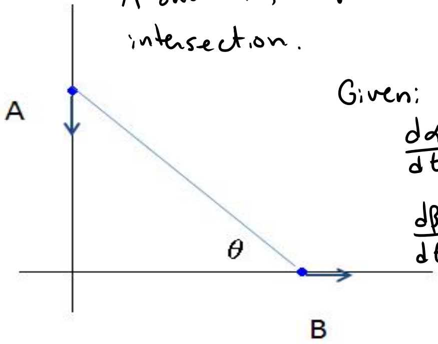
Example

Pedestrians A and B are walking on straight streets that meet at right angles. A approaches the intersection at 2m/sec, and B moves away from the intersection at 1m/sec. At what rate is the angle θ changing when A is 10m from the intersection and B is 20 m from the intersection?

Let α and β be the distances of persons A and B, respectively, from the intersection.

Rates:

$\frac{d\alpha}{dt}$, $\frac{d\beta}{dt}$, $\frac{d\theta}{dt}$



Given:

$$\frac{d\alpha}{dt} = -2 \ \frac{m}{sec}$$

$$\frac{d\beta}{dt} = 1 \ \frac{m}{sec}$$

Q: $\frac{d\theta}{dt} = ?$ when $\alpha = 10m$ and $\beta = 20m$

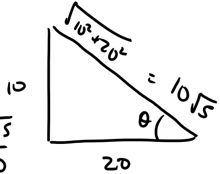
From the diagram $\tan\theta = \frac{\alpha}{\beta}$

Take $\frac{d}{dt}$ of both sides

$$\sec^2\theta \, \frac{d\theta}{dt} = \frac{\frac{d\alpha}{dt}\beta - \frac{d\beta}{dt}\alpha}{\beta^2}$$

$$\frac{d\theta}{dt} = \frac{\frac{d\alpha}{dt}\beta - \frac{d\beta}{dt}\alpha}{\beta^2 \sec^2\theta}$$

when $\alpha = 10$ $\beta = 20$



$\sec\theta = \frac{10\sqrt{5}}{20}$

When $\alpha = 10\,m$ and $\beta = 20\,m$

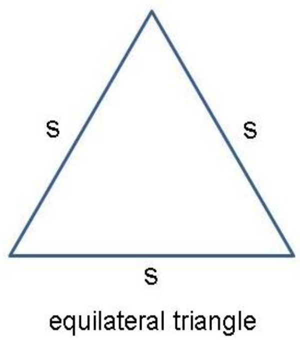
$$\frac{d\theta}{dt} = \frac{-2\frac{m}{sec}\cdot 20m - 1\frac{m}{sec}\cdot 10m}{(20m)^2\left(\frac{\sqrt{5}}{2}\right)^2} = \frac{-50\frac{m^2}{sec}}{500\ m^2}$$

$$\frac{d\theta}{dt} = \frac{-1}{10}\ \frac{1}{sec}$$

θ is decreasing at a rate of $\frac{1}{10}$ radians per second.

Your Turn

The sides of an equilateral triangle are increasing at a constant rate of 2 cm/min.



Determine the rate at which the area is increasing when
(a) the sides are 6 cm long
(b) the area is $8\sqrt{3}$ cm^2