Sept 23 Math 2253H sec. 05H Fall 2014

Section 3.1: Maximums and Minimums

Definition: A **critical number** of a function f is a number c in its domain such that either

$$f'(c) = 0 \quad \text{or} \quad f'(c) \text{ does not exist.}$$

Theorem: If f has a local extremum at c, then c is a critical number of f.

Critical numbers are also called critical points.

Example

Find all of the critical numbers of the function.

The domain of f is all reals.

(a) $f(x) = x^4 - 2x^2 + 5$

$f'(x) = 4x^3 - 4x = 4x(x^2-1) = 4x(x-1)(x+1)$

We need to determine when $f'(x)=0$ and when $f'(x)$ DNE.

$f'(x)$ is undefined never.

$f'(x)=0$ if $4x(x-1)(x+1) = 0$ i.e. $x=0, 1,$ or -1.

f has 3 critical numbers, 0, 1, and -1.

The domain is all reals.

(b) $g(t) = t^{1/5}(12-t)$

$$g(t) = 12\, t^{1/5} - t^{6/5}$$

$$g'(t) = 12\left(\frac{1}{5}\right) t^{-4/5} - \frac{6}{5}\, t^{1/5}$$

$$= \frac{12}{5\, t^{4/5}} - \frac{6\, t^{1/5}}{5} \cdot \frac{t^{4/5}}{t^{4/5}}$$

$$= \frac{12}{5\, t^{4/5}} - \frac{6t}{5\, t^{4/5}}$$

$$\Rightarrow \quad g'(t) = \frac{12-6t}{5\, t^{4/5}}$$

Simplify to get one fraction

$$g'(t) = 0 \implies 12 - 6t = 0 \implies t = 2$$

$$g'(t) \text{ DNE} \implies 5t^{4/5} = 0 \implies t = 0$$

g has two critical numbers, 0 and 2.

The domain is $\{x \mid x \neq 0\}$.

(c) $h(x) = x + \frac{1}{x}$

$$h(x) = x + x^{-1}$$

$$h'(x) = 1 - x^{-2} = 1 - \frac{1}{x^2} = \frac{x^2}{x^2} - \frac{1}{x^2}$$

$$h'(x) = \frac{x^2 - 1}{x^2}$$

$h'(x) = 0 \Rightarrow x^2 - 1 = 0 \Rightarrow x = 1$ or $x = -1$

$h'(x)$ DNE $\Rightarrow x^2 = 0 \Rightarrow x = 0$ this is not in the domain

h has two critical numbers, 1 and -1.

Example

Find the absolute maximum and absolute minimum values of the function on the closed interval.

(a) $f(x) = 1+27x-x^3$, on $[0,4]$

f is continuous on [0,4] and [0,4] is closed. The EVT applies.

The extrema occur at an endpoint or somewhere in between.

If it occurs inside, it does so at a critical number.

Find critical no: $f'(x) = 27 - 3x^2 = 3(9-x^2)$

$f'(x)$ DNE never, $f'(x) = 0 \Rightarrow x = 3$ or $x = -3$.

f has one critical no. inside $[0,4]$, namely 3.

Compare f at the end points and the critical numbers:

$f(0) = 1 + 27 \cdot 0 - 0^3 = 1 \leftarrow$ minimum

$f(3) = 1 + 27 \cdot 3 - 3^3 = 55 \leftarrow$ maximum

$f(4) = 1 + 27 \cdot 4 - 4^3 = 45$

The absolute minimum value of f on $[0,4]$ is $1 = f(0)$.

The absolute maximum value of f on $[0,4]$ is $55 = f(3)$.

(b) $g(t) = t^{1/5}(12-t)$, on $[-1, 1]$

g is continuous + $[-1,1]$ is closed. The EVT applies.

From before, g has one critical number inside $[-1,1]$, 0.

Compare g at the ends and the critical number:

$$g(-1) = (-1)^{1/5}(12-(-1)) = -13$$

$$g(0) = (0)^{1/5}(12-0) = 0$$

$$g(1) = (1)^{1/5}(12-1) = 11$$

The absolute maximum value of g on $[-1,1]$

is $\quad 11 = g(1)$.

The absolute minimum value of g on $[-1,1]$

is $\quad -13 = g(-1)$.

The EVT holds.

(c) $f(x) = \cos x - \dfrac{x}{2}, \quad$ on $\quad \left[-\dfrac{\pi}{2}, \dfrac{\pi}{2}\right]$

$f'(x) = -\sin x - \frac{1}{2}$ $\qquad$ $f'(x)$ DNE never

$f'(x) = 0 \implies -\sin x - \frac{1}{2} = 0 \implies \sin x = \frac{-1}{2}$

There is one critical number in $\left[\frac{-\pi}{2}, \frac{\pi}{2}\right]$.

$x = \frac{-\pi}{6}$.

Compare f at the ends and critical no.

$$f\left(\frac{-\pi}{2}\right) = \cos\left(\frac{-\pi}{2}\right) - \frac{-\pi/2}{2} = \frac{\pi}{4}$$

$$f\left(\frac{-\pi}{6}\right) = \cos\left(\frac{-\pi}{6}\right) - \frac{-\pi/6}{2} = \frac{\sqrt{3}}{2} + \frac{\pi}{12} \leftarrow \text{maximum}$$

$$f\left(\frac{\pi}{2}\right) = \cos\left(\frac{\pi}{2}\right) - \frac{\pi/2}{2} = \frac{-\pi}{4} \leftarrow \text{minimum}$$

The absolute max of f on $[-\pi/2, \pi/2]$ is

$$f\left(\frac{-\pi}{6}\right) = \frac{\sqrt{3}}{2} + \frac{\pi}{2}.$$

The absolute min of f on $[-\pi/2, \pi/2)$

is $f\left(\frac{\pi}{2}\right) = \frac{-\pi}{4}$.