Sept 29 Math 2253H sec. 05H Fall 2014

Section 3.2: The Mean Value Theorem

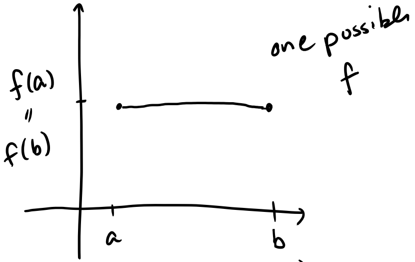
Rolle's Theorem: Let f be a function that is

i continuous on the closed interval $[a, b]$,

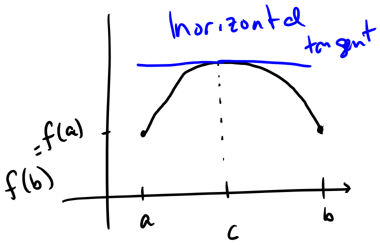
ii differentiable on the open interval (a, b), and

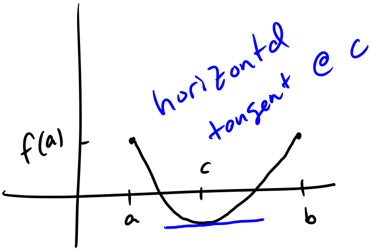
iii such that $f(a) = f(b)$.

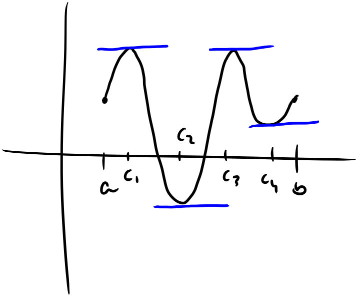
Then there exists a number c in (a, b) such that $f'(c) = 0$.

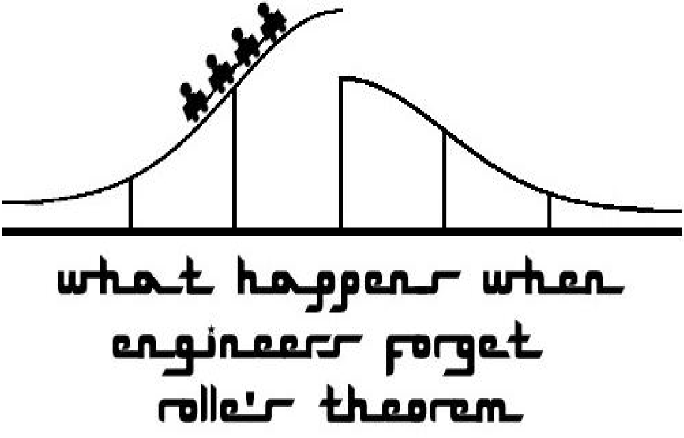


here every x in (a,b) can be our c.








what happens when
engineers forget
rolle's theorem

Example

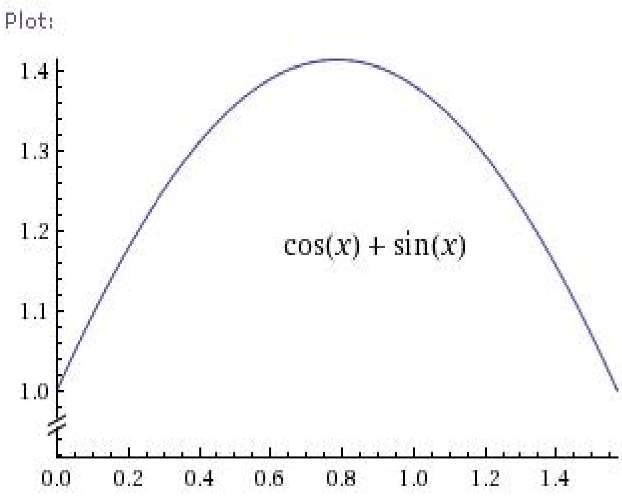
Show that the function $f(\theta) = \cos\theta + \sin\theta$ has at least one point c in $\left[0, \frac{\pi}{2}\right]$ such that $f'(c) = 0$.

$f(\theta)$ is continuous @ all reals, so it is on $\left[0, \frac{\pi}{2}\right]$.

$f'(\theta) = -\sin\theta + \cos\theta$ is well defined everywhere. So f is differentiable on $\left(0, \frac{\pi}{2}\right)$.

$$\left.\begin{aligned} f(0) &= \cos(0) + \sin(0) = 1 + 0 = 1 \\ f\left(\tfrac{\pi}{2}\right) &= \cos\left(\tfrac{\pi}{2}\right) + \sin\left(\tfrac{\pi}{2}\right) = 0 + 1 = 1 \end{aligned}\right\} \Rightarrow f(0) = f\left(\tfrac{\pi}{2}\right)$$

By Rolle's theorem, there must be some c in $(0, \pi/2)$ such that $f'(c) = 0$.



Figure

Example

Show that the polynomial $f(x) = x^3 + x - 1$ has exactly one real root.

Note that $f(0) = -1$ and $f(1) = 1+1-1 = 1$.

Since f is a polynomial, hence continuous, The intermediate value theorem guarantees that $f(c) = 0$ for some c between 0 and 1. i.e. f has at least one real root.

To show it has exactly one real root, let's suppose it has 2 roots - one at x_1

and another at x_2 (assume $x_1 < x_2$).

Note f is continuous on the interval $[x_1, x_2]$.

As a polynomial, f is differentiable on (x_1, x_2).

Moreover, $f(x_1) = 0 = f(x_2)$

By Rolle's theorem, there exist c between x_1 and x_2 such that $f'(c) = 0$.

Now, $f'(x) = 3x^2 + 1$ so $f'(c) = 3c^2 + 1$

giving

$$3c^2+1=0 \Rightarrow c^2 = \frac{-1}{3}$$

which can't be for any real c.

The assumption $f(x_1) = f(x_2) = 0$ must be false.

That is, f has at most one root.

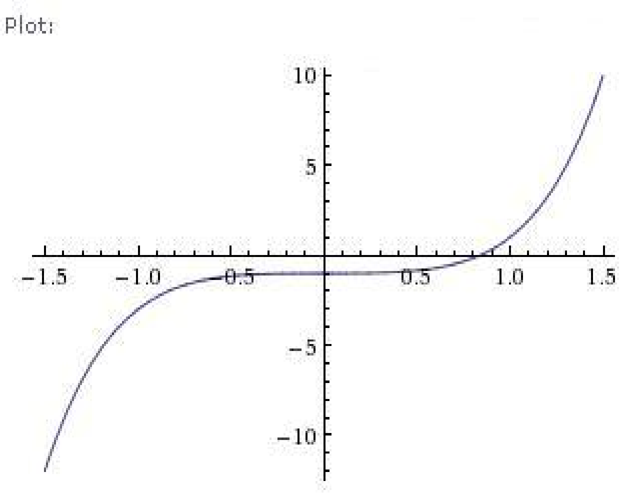


Figure: $y = x^3 + x - 1$

The Mean Value Theorem

Suppose f is a function that satisfies

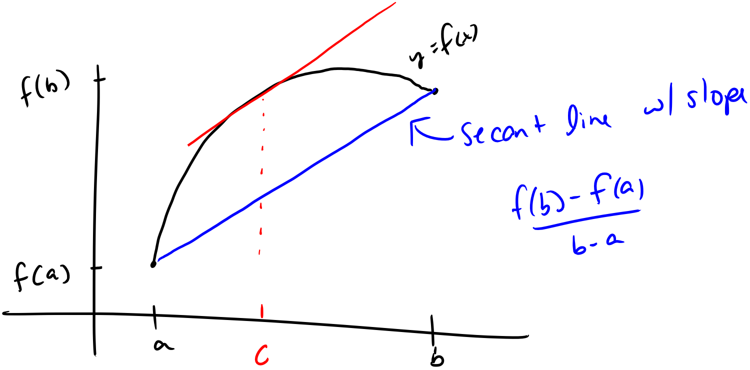
i f is continuous on the closed interval $[a, b]$, and
ii f is differentiable on the open interval (a, b).

Then there exists a number c in (a, b) such that

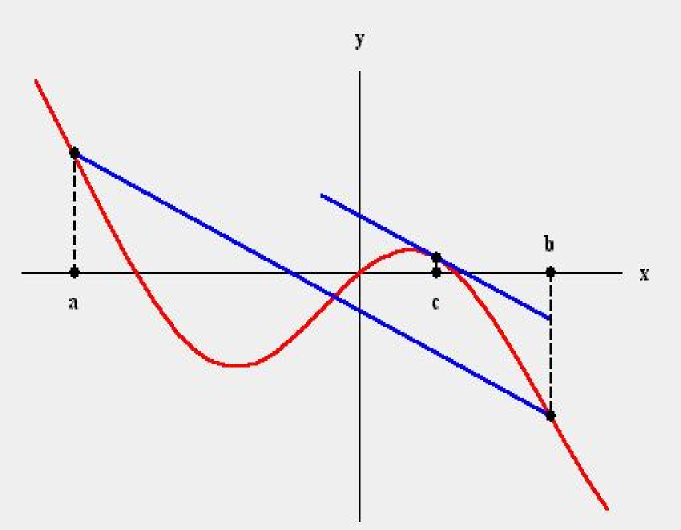
$$f'(c) = \frac{f(b) - f(a)}{b - a}, \quad \text{equivalently} \quad f(b) - f(a) = f'(c)(b - a).$$

$$\frac{f(b) - f(a)}{b - a} \qquad \frac{\Delta y \text{ between end points}}{\Delta x \text{ between end points}}$$

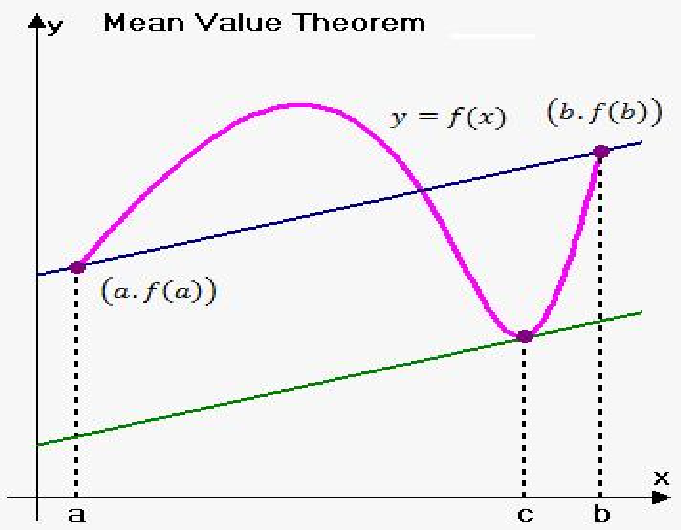
Slope of the secant line



@ $(c, f(c))$ the tangent line is parallel to the secant line.



Figure



Figure

Figure: Celebration of the MVT in Beijing.

Example

Verify that the function satisfies the hypotheses of the Mean Value Theorem on the given interval. Then find all values of c that satisfy the conclusion of the MVT.

$$f(x) = x^3 - 2x, \quad [-2, 2]$$

As a polynomial, f is continuous and differentiable everywhere. So

i) f is continuous on $[-2,2]$ and

ii) f is differentiable on $(-2,2)$.

$f(2) = 2^3 - 2\cdot 2 = 8-4 = 4$, $f(-2) = (-2)^3 - 2(-2) = -8+4 = -4$

So $\frac{f(b)-f(a)}{b-a} = \frac{f(2)-f(-2)}{2-(-2)} = \frac{4+4}{4} = 2$

$f'(x) = 3x^2 - 2 \Rightarrow f'(c) = 3c^2 - 2$

Set $f'(c)$ to $\frac{f(b)-f(a)}{b-a}$

$3c^2 - 2 = 2 \Rightarrow 3c^2 = 4 \Rightarrow c^2 = \frac{4}{3}$

$c = \frac{\pm 2}{\sqrt{3}}$ both are between -2 and 2

So both c-values satisfy the conclusion of the MVT.