Sept 2 Math 2253H sec. 05H Fall 2014

Section 2.1: Derivatives and Rates of Change

Definition: The tangent line to the curve $y = f(x)$ at the point $P(a, f(a))$ is the line through P with slope

$$m = \lim_{x \to a} \frac{f(x) - f(a)}{x - a}$$

provided this limit exists.

If the limit doesn't exist, the graph has no tangent line @ $(a, f(a))$.

When m exists, the tangent line has equation

$$y = m(x-a) + f(a)$$

Example:

Find the equation of the line tangent to the graph of f at the indicated point.

$f(x) = \sqrt{x+1}$ at $(3, 2)$

Find the slope m

$m = \frac{1}{4}$ the point is $(3,2)$

the line is

$y - 2 = \frac{1}{4}(x-3)$

$$\boxed{y = \frac{1}{4}x + \frac{5}{4}}$$

$$m = \lim_{x\to 3} \frac{f(x) - f(3)}{x-3}$$

$$= \lim_{x\to 3} \frac{\sqrt{x+1} - 2}{x-3} \cdot \frac{\sqrt{x+1}+2}{\sqrt{x+1}+2}$$

$$= \lim_{x\to 3} \frac{x+1-4}{(x-3)(\sqrt{x+1}+2)}$$

$$= \lim_{x\to 3} \frac{x-3}{(x-3)(\sqrt{x+1}+2)}$$

$$= \lim_{x\to 3} \frac{1}{\sqrt{x+1}+2} = \frac{1}{4}$$

Example Illustrated

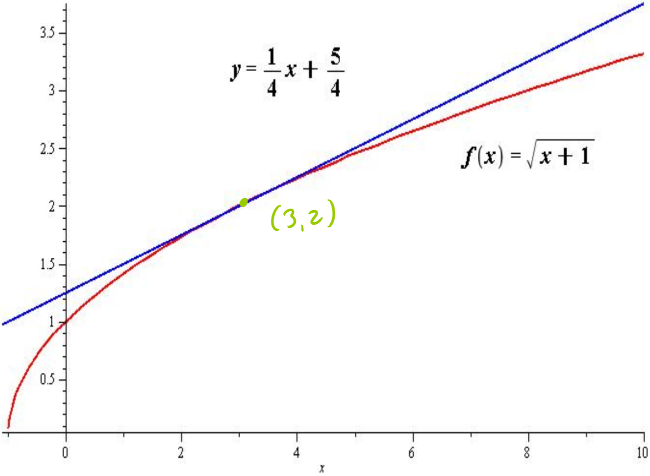


Figure: Note that near the point $(3, 2)$, the points on the line are very close to the points on the curve.

Velocities

Suppose $s = f(t)$ represents the position of particle moving along a straight line (e.g. the x-axis). Consider a time interval between $t = a$ and $t = a + h$.

$$\text{displacement (position change)} = f(a+h) - f(a) = \Delta s$$

$$\text{time change} = a + h - a = h = \Delta t$$

$$\text{average velocity} = \frac{f(a+h) - f(a)}{h} = \frac{\Delta s}{\Delta t}$$

The **velocity** (instantaneous velocity) of the particle at time $t = a$ is

$$v(a) = \lim_{h \to 0} \frac{f(a+h) - f(a)}{h} = \lim_{\Delta t \to 0} \frac{\Delta s}{\Delta t}$$

Rate of Change of a Function

If $y = f(x)$, then the **rate of change**[1] **of** y **with respect to** x at $x = x_1$ is

$$\begin{aligned} \text{rate of change} &= \lim_{\Delta x \to 0} \frac{\Delta y}{\Delta x} \\ &= \lim_{x \to x_1} \frac{f(x) - f(x_1)}{x - x_1} \\ &= \lim_{h \to 0} \frac{f(x_1 + h) - f(x_1)}{h} \end{aligned}$$

If

$x = x_1 + h$

$\Rightarrow h = x - x_1$

$x \to x_1 \Rightarrow h \to 0$

This is also called the **derivative** of y with respect to x at $x = a$ and is denoted by

$$f'(a). \quad \text{(Read "f prime of a".)}$$

for $a = x_1$

[1] a.k.a. *instantaneous rate of change*

Example

Ex. Each limit repesents the derivative of some function f at some number a. Identify such a function and such a number.

(a) $\lim_{h \to 0} \frac{\sqrt[4]{16+h}-2}{h}$

$\lim_{h \to 0} \frac{f(a+h)-f(a)}{h}$

Need $f(a+h) = \sqrt[4]{16+h} \Rightarrow a+h = 16+h$

$a = 16$

$f(x) = \sqrt[4]{x}$

$f(a) = f(16) = \sqrt[4]{16} = 2$ ✓

$\boxed{\text{so } f(x) = \sqrt[4]{x} \text{ and } a = 16}$

(b) $\lim_{x \to \frac{\pi}{4}} \dfrac{\tan x - 1}{x - \frac{\pi}{4}}$

Match to

$$\lim_{x \to a} \frac{f(x) - f(a)}{x - a}$$

$a = \frac{\pi}{4}$ $\qquad$ $f(x) = \tan x$

$f(a) = f\left(\frac{\pi}{4}\right) = \tan\left(\frac{\pi}{4}\right) = 1$ ✓

$$\boxed{\text{So } f(x) = \tan x \text{ and } a = \frac{\pi}{4}.}$$

Section 2.2: The Derivative as a Function

If $f(x)$ is a function, then the set of numbers $f'(a)$ for various values of a should define a new function.

alternative formulation.

Let $f(x)$ be a function. Define the new function

$$f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h} = \lim_{z \to x} \frac{f(z) - f(x)}{z - x}$$

called the **derivative** of f. The domain of this new function is the set $\{x | x \text{ is in the domain of } f\text{, and } f'(x) \text{ exists}\}$.

Example

Identify the domain of f. Determine $f'(x)$ and identify the domain of f'.

$$f(x) = \frac{2}{x}$$

Domain of f is $\{x \mid x \neq 0\}$.

$$f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h}$$

$$= \lim_{h \to 0} \frac{\frac{2}{x+h} - \frac{2}{x}}{h}$$

$$= \lim_{h \to 0} \frac{\frac{2x}{x(x+h)} - \frac{2(x+h)}{x(x+h)}}{h}$$

$$= \lim_{h\to 0} \frac{2x - 2(x+h)}{hx(x+h)}$$

$$= \lim_{h\to 0} \frac{2x - 2x - 2h}{hx(x+h)} = \lim_{h\to 0} \frac{-2h}{hx(x+h)}$$

$$= \lim_{h\to 0} \frac{-2}{x(x+h)} = \frac{-2}{x(x+0)} = \frac{-2}{x^2}$$

$$\boxed{f'(x) = \frac{-2}{x^2} \quad \text{w/ domain} \quad \{x \mid x \neq 0\}}$$

Example Continued...

Use the results to find the equation of the line tangent to the graph of $y = 2/x$ at the point $(2, 1)$.

The tangent line has slope $m = f'(2)$.

From the last example $f'(2) = \frac{-2}{(2^2)} = \frac{-1}{2}$

The line is $y - 1 = \frac{-1}{2}(x-2)$

$$\boxed{y = \frac{-1}{2}x + 2}$$

Example

Find all points on the graph of $y = 2/x$ at which the rate of change of y with respect to x is -4. Are there any points at which the rate of change of y with respect to x is 1?

The rate of change of y is $f'(x)$.

Set $f'(x)$ equal to -4:

$$\frac{-2}{x^2} = -4 \implies x^2 = \frac{1}{2} \implies x = \frac{1}{\sqrt{2}} \text{ or } x = \frac{-1}{\sqrt{2}}$$

Get y-values: $\frac{2}{1/\sqrt{2}} = 2\sqrt{2}$, $\frac{2}{-1/\sqrt{2}} = -2\sqrt{2}$

There are two points at which the rate of change of y is -4

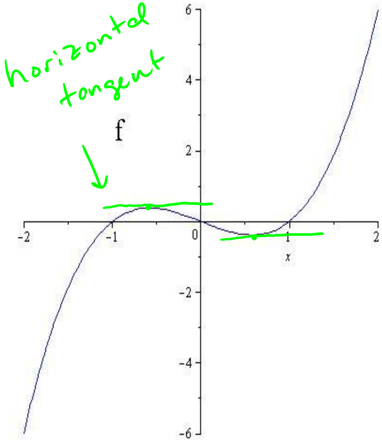
$$\left(\frac{1}{\sqrt{2}}, 2\sqrt{2}\right) \text{ and } \left(-\frac{1}{\sqrt{2}}, -2\sqrt{2}\right).$$

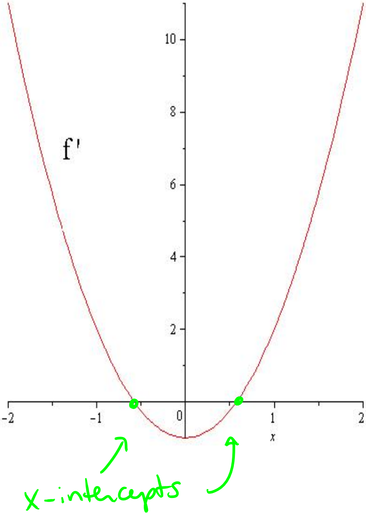
Set $f'(x)$ to 1:

$$\frac{-2}{x^2} = 1 \implies -2 = x^2$$

There are no real number solutions.

How are the functions $f(x)$ and $f'(x)$ related?





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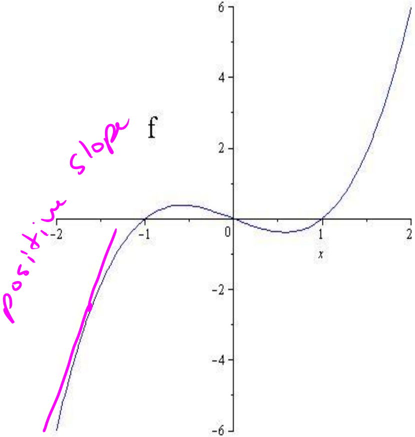

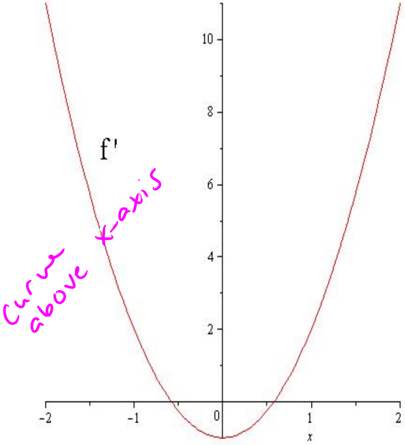

Remarks:

- if $f(x)$ is a function of x, then $f'(x)$ is a new function of x (called the derivative of f)
- The number $f'(a)$ (if it exists) is the slope of the curve of $y = f(x)$ at the point $(a, f(a))$
- this is also the slope of the tangent line to the curve of y at $(a, f(a))$
- ”slope of the curve”, ”slope of the tangent line”, and ”rate of change” are the same concept

Definition: A function f is said to be *differentiable* at a if $f'(a)$ exists. It is called *differentiable* on an open interval I if it is differentiable at each point in I.