Sept 30 Math 2253H sec. 05H Fall 2014

Section 3.2: The Mean Value Theorem

Suppose f is a function that satisfies

i f is continuous on the closed interval $[a,b]$, and

ii f is differentiable on the open interval (a,b).

Then there exists a number c in (a,b) such that

$$f'(c) = \frac{f(b)-f(a)}{b-a}, \quad \text{equivalently} \quad f(b)-f(a) = f'(c)(b-a).$$

That is, for some c in the interval, the tangent line at $(c, f(c))$ is parallel to the secant line through the points $(a, f(a))$ and $(b, f(b))$.

Example

Let f be a function that is differentiable for all real x. Suppose $f(0) = 3$ and $f'(x) \leq 2$ for all $0 \leq x \leq 10$. What is the maximum possible value of $f(10)$?

f is continuous on $[0,10]$

f is differentiable on $(0,10)$

By the MVT, there exists c in $(0,10)$ such that

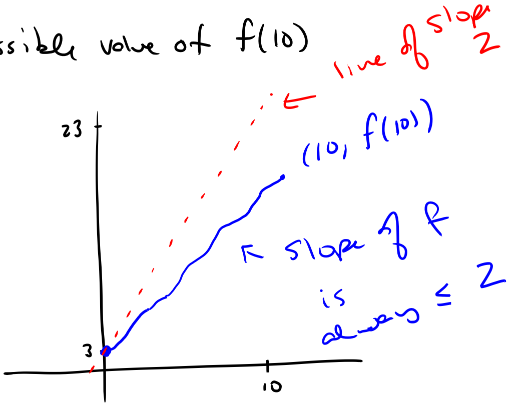
$$f'(c) = \frac{f(10) - f(0)}{10 - 0}$$

$$f'(c) = \frac{f(10) - 3}{10} \Rightarrow f(10) - 3 = 10 f'(c)$$

$$\Rightarrow \quad f(10) = 10 f'(c) + 3 \qquad f'(c) \le 2$$

$$\Rightarrow \quad f(10) \le 10(2) + 3 = 23$$

So the maximum possible value of $f(10)$ is 23.



Important Consequence of the MVT

Theorem: If $f'(x) = 0$ for all x in an interval (a, b), then f is constant on (a, b).

Corollary: If $f'(x) = g'(x)$ for all x in an interval (a, b), then $f - g$ is constant on (a, b). In other words,

$$f(x) = g(x) + C \quad \text{where } C \text{ is some constant.}$$

Suppose $f'(x) = 0$ for x in (a,b). Let $a < x_1 < x_2 < b$. f is continuous on $[x_1, x_2]$ and differentiable on (x_1, x_2). By the MVT there exists c in (x_1, x_2) such that

$$f(x_2) - f(x_1) = f'(c)(x_2 - x_1)$$

$$f(x_2) - f(x_1) = 0 \Rightarrow$$

$$\Rightarrow \quad f(x_2) = f(x_1)$$

But x_2 and x_1 can be any numbers in (a,b).

That is, "$f(x) =$ some number" for all x in (a,b).

Examples

Find all possible functions $f(x)$ that satisfy the condition

(a) $f'(x) = \cos x$ on $(-\infty, \infty)$

Recall $\frac{d}{dx} \sin x = \cos x$

So $f(x) = \sin x + C$

where C is any constant

(b) $f'(x) = 2x$ on $(-\infty, \infty)$

$$\frac{d}{dx} x^2 = 2x$$

So $f(x) = x^2 + C$

for arbitrary constant C

Find all possible functions $h(t)$ that satisfy the condition

(c) $h'(t) = \sec^2 t$ on $\left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$

$$\frac{d}{dt} \tan t = \sec^2 t$$

So $h(t) = \tan t + C$

for arbitrary constant C

Section 3.3: Derivatives and the Shapes of Graphs

If f is differentiable on a domain, the derivative f' gives information about f. f'' also gives information about f if it exists.

Theorem: (Increasing/Decreasing test)

- If $f'(x) > 0$ on an interval, then f is increasing on that interval.
- If $f'(x) < 0$ on an interval, then f is decreasing on that interval.

Example

Determine the intervals on which f is increasing and the intervals on which f is decreasing if f has the following derivative

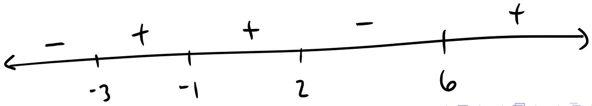
$$f'(x) = 2(x+3)(x+1)^2(x-2)(x-6)$$

To find where $f'(x) > 0$ or $f'(x) < 0$ determine where the sign changes

$f'(x) = 0 \Rightarrow 0 = 2(x+3)(x+1)^2(x-2)(x-6)$

$\Rightarrow x = -3, -1, 2, 6$

Sign analysis of f'



$f'(-4)$ $(-)(+)(-)(-)$, $f(-2)$ $(+)(+)(-)(-)$

$f'(0)$ $(+)(+)(-)(-)$, $f(3)$ $(+)(+)(+)(-)$

$f(7)$ $(+)(+)(+)(+)$

f is increasing on the intervals

$(-3,-1)$, $(-1,2)$, and $(6,\infty)$

f is decreasing on the intervals

$(-\infty,-3)$ and $(2,6)$

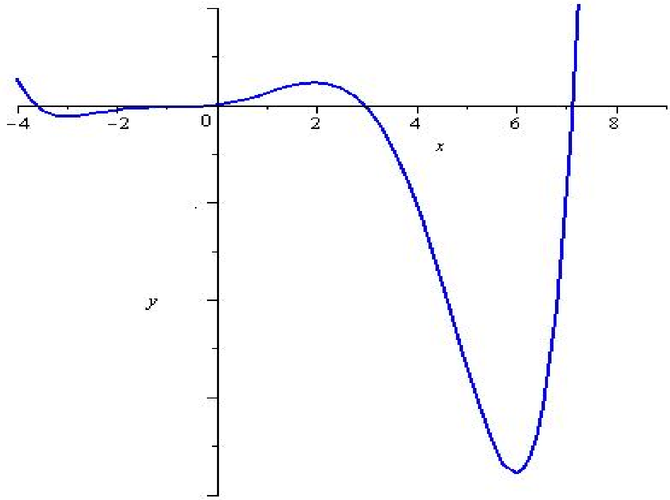


Figure: A function f with derivative $f'(x) = 2(x+3)(x+1)^2(x-2)(x-6)$

Theorem: First derivative test for local extrema

Let f be continuous and suppose that c is a critical number of f.

- If f' changes from positive to negative at c, then f has a local maximum at c.
- If f' changes from negative to positive at c, then f has a local minimum at c.
- If f' does not change signs at c, then f does not have a local extremum at c.

Note: we read from left to right as usual when looking for a sign change.

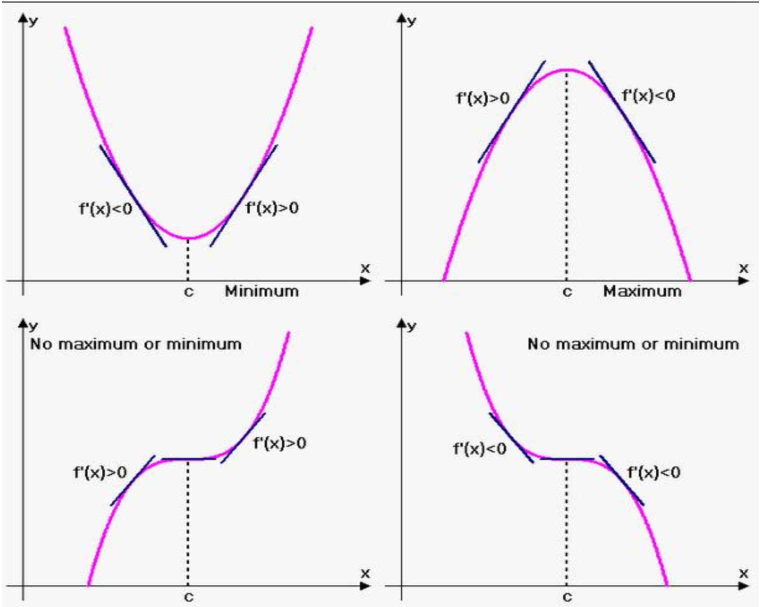


Figure: First derivative test

Example

Find all the critical points of the function and classify each one as a local maximum, a local minimum, or neither.

$$s(t) = t^4 - 8t^3 + 10t^2 - 4$$

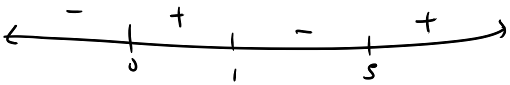
Crit #: $s'(t) = 4t^3 - 24t^2 + 20t$

$s'(t)$ is never undefined

$s'(t) = 0 \Rightarrow$
$$\begin{aligned} 0 &= 4t^3 - 24t^2 + 20t \\ &= 4t(t^2 - 6t + 5) \\ &= 4t(t-1)(t-5) \end{aligned}$$

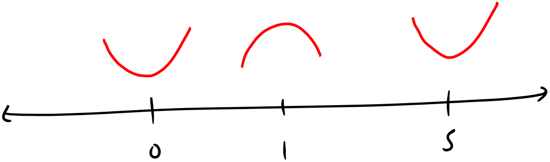
$\Rightarrow t = 0, 1,$ or 5

Sign analysis on $s'(t)$



$s'(-1)$ $(-)(-)(-)$, $s'(\frac{1}{2})$ $(+)(-)(-)$

$s'(2)$ $(+)(+)(-)$, $s'(6)$ $(+)(+)(+)$



f takes local minimums @ 0 and 5

takes a local maximum @ $t = 1$