Sept 4 Math 2253H sec. 05H Fall 2014

Section 2.2: The Derivative as a Function

Definition: Let $f(x)$ be a function. Define the new function

$$f'(x) = \lim_{h\to 0} \frac{f(x+h)-f(x)}{h} = \lim_{z\to x} \frac{f(z)-f(x)}{z-x}$$

called the **derivative** of f. The domain of this new function is the set $\{x | x$ is in the domain of f, and $f'(x)$ exists$\}$.

Remarks:

- if $f(x)$ is a function of x, then $f'(x)$ is a new function of x (called the derivative of f)
- The number $f'(a)$ (if it exists) is the slope of the curve of $y = f(x)$ at the point $(a, f(a))$
- this is also the slope of the tangent line to the curve of y at $(a, f(a))$
- "slope of the curve", "slope of the tangent line", and "rate of change" are the same concept

Definition: A function f is said to be *differentiable* at a if $f'(a)$ exists. It is called *differentiable* on an open interval I if it is differentiable at each point in I.

Notation

If $y = f(x)$, the following notation are interchangeable:

$$f'(x) = y'(x) = y' = \frac{dy}{dx} = \frac{df}{dx} = \frac{d}{dx}f(x) = Df(x) = D_x f(x)$$

$$\text{Leibniz Notation:} \qquad \lim_{\Delta x \to 0} \frac{\Delta y}{\Delta x} = \frac{dy}{dx}$$

You can think of $\dfrac{d}{dx}$ is an "operator."

It acts on a function to produce a new function—its derivative.

Example

Find $\frac{dy}{dx}$ if $y = \sqrt{x}$ and determine it's domain.

Domain of y is $\{x \mid x \geq 0\}$.

$$\frac{dy}{dx} = \lim_{z \to x} \frac{\sqrt{z} - \sqrt{x}}{z - x}$$

$$= \lim_{z \to x} \frac{\sqrt{z} - \sqrt{x}}{z - x} \cdot \frac{\sqrt{z} + \sqrt{x}}{\sqrt{z} + \sqrt{x}}$$

$$= \lim_{z \to x} \frac{z - x}{(z - x)(\sqrt{z} + \sqrt{x})}$$

$$= \lim_{z \to x} \frac{1}{\sqrt{z} + \sqrt{x}} = \frac{1}{\sqrt{x} + \sqrt{x}} = \frac{1}{2\sqrt{x}}$$

$$\boxed{\frac{dy}{dx} = \frac{1}{2\sqrt{x}}}$$

The domain of $\frac{dy}{dx}$ is $\{x \mid x > 0\}$.

* y is not differentiable @ zero. *

Failure to be Differentiable

Show that $y = |x|$ is not differentiable at zero.

By definition

$$|h| = \begin{cases} h, & h \geq 0 \\ -h, & h < 0 \end{cases}$$

If it exists, then . .

$$y'(0) = \lim_{h \to 0} \frac{|0+h| - |0|}{h} = \lim_{h \to 0} \frac{|h|}{h}$$

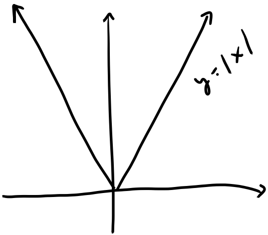
we'll consider the one sided limits

$$\lim_{h \to 0^-} \frac{|h|}{h} = \lim_{h \to 0^-} \frac{-h}{h} = \lim_{h \to 0^-} -1 = -1$$

$$\lim_{h\to 0^+} \frac{|h|}{h} = \lim_{h\to 0^+} \frac{h}{h} = \lim_{h\to 0^+} 1 = 1$$

$$\lim_{h\to 0} \frac{|h|}{h} \text{ DNE}$$ (one sided limits disagree)

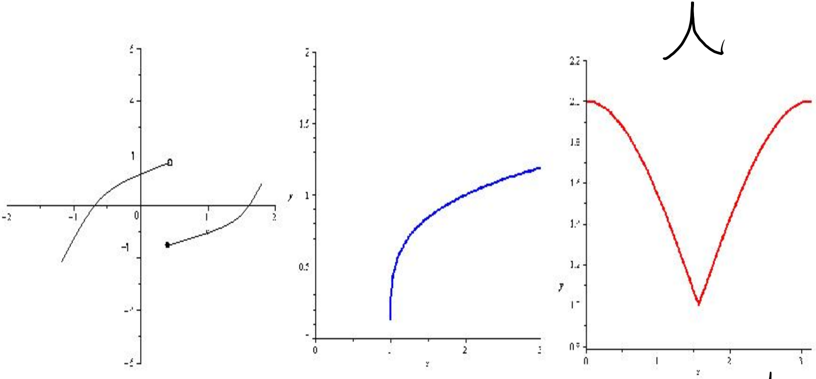
So $y=|x|$ is not differentiable at zero.



for $x>0$ $\quad \frac{dy}{dx}=1$

for $x<0$ $\quad \frac{dy}{dx}=-1$

Failure to be differentiable: Discontinuity, Vertical tangent, or Corner/Cusp



jump

Vertical tangent

corner/cusp

Theorem

Differentiability implies continuity.

That is, if f is differentiable at a, then f is continuous at a. The corner example shows that the converse of this is not true!

Higher Order Derivatives:

First derivative: $\dfrac{dy}{dx} = y' = f'(x)$

Second derivative: $\dfrac{d}{dx}\dfrac{dy}{dx} = \dfrac{d^2y}{dx^2} = y'' = f''(x)$

Third derivative: $\dfrac{d}{dx}\dfrac{d^2y}{dx^2} = \dfrac{d^3y}{dx^3} = y''' = f'''(x)$

parentheses are required

Fourth derivative: $\dfrac{d}{dx}\dfrac{d^3y}{dx^3} = \dfrac{d^4y}{dx^4} = y^{(4)} = f^{(4)}(x)$

n^{th} derivative: $\dfrac{d}{dx}\dfrac{d^{n-1}y}{dx^{n-1}} = \dfrac{d^ny}{dx^n} = y^{(n)} = f^{(n)}(x)$

Section 2.3: Some Differentiation Rules

The derivative of a constant function is zero.

$$\frac{d}{dx}c = 0$$

The derivative of the identity function is one.

$$\frac{d}{dx}x = 1$$

For positive integer n, the **power rule** says

$$\frac{d}{dx}x^n = nx^{n-1}$$

Evaluate Each Derivative

(a) $\dfrac{d}{dx} 3\pi = 0$ $\quad$ 3π is constant

(b) $\dfrac{d}{dx} x^9 = 9x^{9-1} = 9x^8$

More Derivative Rules

Assume f and g are differentiable functions and c is a constant.
Constant multiple rule:

$$\frac{d}{dx}cf(x) = cf'(x)$$

Sum rule:

$$\frac{d}{dx}(f(x) + g(x)) = f'(x) + g'(x)$$

Difference rule:

$$\frac{d}{dx}(f(x) - g(x)) = f'(x) - g'(x)$$

Example: Evaluate Each Derivative

(a) $\dfrac{d}{dx}(x^4-3x^2) = \dfrac{d}{dx}x^4 - \dfrac{d}{dx}3x^2$

$= \dfrac{d}{dx}x^4 - 3\dfrac{d}{dx}x^2$

$= 4x^{4-1} - 3(2x^{2-1}) = 4x^3 - 6x$

(b) $\dfrac{d}{dx}(2x^3+3x^2-12x+1) = \dfrac{d}{dx}2x^3 + \dfrac{d}{dx}3x^2 - \dfrac{d}{dx}12x + \dfrac{d}{dx}1$

$= 2\dfrac{d}{dx}x^3 + 3\dfrac{d}{dx}x^2 - 12\dfrac{d}{dx}x + \dfrac{d}{dx}1$

$= 2(3x^2) + 3(2x) - 12(1) + 0 = 6x^2 + 6x - 12$

Example

If $f(x) = 2x^3 + 3x^2 - 12x + 1$, find all points on the graph of f at which the slope of the graph is zero.

We want to find x-values at which $f'(x) = 0$.

$$0 = f'(x) = 6x^2 + 6x - 12 = 6(x^2 + x - 2)$$

$$\Rightarrow \quad 0 = 6(x-1)(x+2)$$

$$\Rightarrow \quad x = 1 \quad \text{or} \quad x = -2$$

y-values

$$f(1) = 2(1)^3 + 3(1)^2 - 12(1) + 1 = -6$$

$$f(-2) = 2(-2)^3 + 3(-2)^2 - 12(-2) + 1$$

$$= -16 + 12 + 24 + 1 \qquad = 21$$

There are two points at which the slope of the graph is zero

$(1, -6)$ and $(-2, 21)$.

Even More Rules!

Assume f and g are differentiable functions and n is any real number

Product rule:

$$\frac{d}{dx} f(x)g(x) = f'(x)g(x) + f(x)g'(x) = f'g + g'f$$

Quotient rule:

$$\frac{d}{dx}\frac{f(x)}{g(x)} = \frac{f'(x)g(x) - f(x)g'(x)}{(g(x))^2} = \frac{f'g - g'f}{g^2}$$

General Power rule:

$$\frac{d}{dx} x^n = nx^{n-1}$$

Example

Evaluate $f'(x)$ two ways where $f(x) = x^2$.

$* \ \frac{d}{dx}(x \cdot x) \neq \left(\frac{d}{dx}x\right) \cdot \left(\frac{d}{dx}x\right)$

$1 \cdot 1 = 1 \ *$

Using the power rule.

$$f'(x) = 2x^{2-1} = 2x$$

Write $f(x) = xx$ (x times x), and use the product rule.

$$f'(x) = \left(\frac{d}{dx}x\right)x + x\left(\frac{d}{dx}x\right)$$

$$= 1x + x \cdot 1 = x + x = 2x$$

Example

Find $\frac{dy}{dx}$ given

$$y = (2x^3 + x)(x^\pi - 14)$$

Using the product rule

$$\frac{dy}{dx} = (6x^2 + 1)(x^\pi - 14) + (2x^3 + x)(\pi x^{\pi - 1} - 0)$$

$$= (6x^2 + 1)(x^\pi - 14) + \pi x^{\pi - 1}(2x^3 + x)$$

Example

Find $\frac{dy}{dx}$ given

$$y = \frac{x+1}{x^2+2x}$$

$$\frac{dy}{dx} = \frac{(1)(x^2+2x) - (x+1)(2x+2)}{(x^2+2x)^2}$$

$$= \frac{x^2+2x - (2x^2+4x+4)}{(x^2+2x)^2} = \frac{-x^2-2x-4}{(x^2+2x)^2}$$