# Sept 8 Math 2253H sec. 05H Fall 2014

## Section 2.4: Derivatives of Trigonometric Functions

Preliminary results:

$$\lim_{\theta\to 0}\frac{\sin\theta}{\theta}=1 \quad \text{and} \quad \lim_{\theta\to 0}\frac{\cos\theta-1}{\theta}=0$$

We can prove from the definition of the derivative that

$$\frac{d}{dx}\sin(x)=\cos(x) \quad \text{and} \quad \frac{d}{dx}\cos(x)=-\sin(x)$$

$$\frac{d}{dx}\sin(x) = \lim_{h\to 0} \frac{\sin(x+h) - \sin x}{h}$$

$* \sin(A+B) =$

$\sin A \cos B + \sin B \cos A$

$*$

$$= \lim_{h\to 0} \frac{\sin x \cos h + \sin h \cos x - \sin x}{h}$$

$$= \lim_{h\to 0} \frac{\sin x(\cos h - 1) + \cos x \sin h}{h}$$

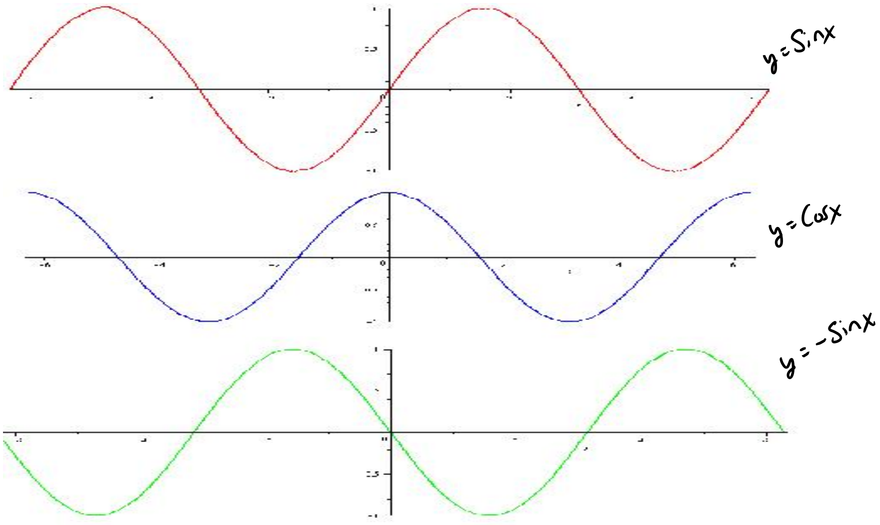
$$= \lim_{h\to 0} \frac{\sin x(\cos h - 1)}{h} + \lim_{h\to 0} \frac{\cos x \sin h}{h}$$

$$= \sin x \lim_{h\to 0} \frac{\cos h - 1}{h} + \cos x \lim_{h\to 0} \frac{\sin h}{h}$$

$$= \sin x\,(0) + \cos x\,(1)$$

$$= \cos x$$

i.e. $$\frac{d}{dx} \sin x = \cos x$$



y = Sinx
y = Cosx
y = -Sinx

Evaluate the derivative.

(a) $\frac{d}{dx}(\sin x + 4\cos x) = \cos x + 4(-\sin x)$

$= \cos x - 4\sin x$

(b) $\frac{d}{d\theta}\theta^4 \sin\theta = 4\theta^3 \sin\theta + \theta^4 \cos\theta$

Use the fact that $\tan x = \sin x / \cos x$ to determine the derivative rule for the tangent.

$$\frac{d}{dx}\tan x = \frac{d}{dx}\left(\frac{\sin x}{\cos x}\right) = \frac{\cos x \cos x - (-\sin x)\sin x}{\cos^2 x}$$

$$= \frac{\cos^2 x + \sin^2 x}{\cos^2 x}$$

$$= \frac{1}{\cos^2 x}$$

$$= \sec^2 x$$

# Six Trig Function Derivatives

$$\frac{d}{dx}\sin x = \cos x, \qquad \frac{d}{dx}\cos x = -\sin x,$$

$$\frac{d}{dx}\tan x = \sec^2 x, \qquad \frac{d}{dx}\cot x = -\csc^2 x,$$

$$\frac{d}{dx}\sec x = \sec x \tan x, \qquad \frac{d}{dx}\csc x = -\csc x \cot x$$

# Example

Find $\frac{dg}{dt}$.

$$g(t) = \sqrt{t} - 2\cot t \;=\; t^{1/2} - 2\cot t$$

$$g'(t) = \frac{1}{2}t^{-1/2} - 2(-\csc^2 t)$$

$$g'(t) = \frac{1}{2\sqrt{t}} + 2\csc^2 t$$

# Example

Find the equation of the line normal to the graph of $y = \sec x$ at the point $(\pi, -1)$.

Get slope of the tangent line:

$$\frac{dy}{dx} = \sec x \tan x$$

Tangent line has slope $m = \left.\frac{dy}{dx}\right|_{x=\pi} = \sec\pi \tan\pi = 0$

The normal line is vertical.

$$\boxed{x = \pi}$$

# Example

Use the limit identity $\lim_{\theta\to 0} \frac{\sin\theta}{\theta} = 1$ to evaluate the indicated limit.

(a) $\lim_{x\to 0} \frac{\sin(2x)}{x}$ $= \lim_{x\to 0} \frac{\sin(2x)}{2x} \cdot 2$

$= 2 \lim_{x\to 0} \frac{\sin(2x)}{2x}$

$= 2(1) = 2$

(b) $\lim\limits_{x \to 0} 3x\csc(4x) = \lim\limits_{x\to 0} \dfrac{3x}{\sin(4x)} = \lim\limits_{x\to 0} \left(\dfrac{\sin(4x)}{3x}\right)^{-1}$

$$= \lim_{x\to 0} \left(\frac{1}{3}\,\frac{\sin(4x)}{x}\right)^{-1}$$

$$= \lim_{x\to 0} \left(\frac{4}{3}\,\frac{\sin(4x)}{4x}\right)^{-1}$$

$$= \left(\frac{4}{3}\cdot 1\right)^{-1} = \frac{3}{4}$$

# Section 2.5: The Chain Rule

Suppose we wish to find the derivative of $f(x) = (x^2 + 2)^2$.

$$f(x) = x^4 + 4x^2 + 4$$

$$f'(x) = 4x^3 + 8x$$

Now suppose we want to differentiate $g(x) = (x^2 + 2)^{10}$. How about $F(x) = \sqrt{x^2 + 2}$?

# Example of Compositions

Find functions $f(u)$ and $g(x)$ such that

$$F(x) = \sqrt{x^2+2} = (f \circ g)(x).$$

$$= f(g(x))$$

$$f(u) = \sqrt{u}$$

$$g(x) = x^2+2$$

Check:

$$f(g(x)) = f(x^2+2)$$

$$= \sqrt{x^2+2}$$

$$= F(x) \quad \checkmark$$

# Example of Compositions

Find functions $f(u)$ and $g(x)$ such that

$$F(x) = \cos\left(\frac{\pi x}{2}\right) = (f \circ g)(x).$$

$f(u) = \cos(u)$

$g(x) = \frac{\pi}{2} x$

Check:

$f(g(x)) = f\left(\frac{\pi}{2} x\right)$

$= \cos\left(\frac{\pi}{2} x\right).$

$= F(x)$ ✓

# Theorem: Chain Rule

Suppose $g$ is differentiable at $x$ and $f$ is differentiable at $g(x)$. Then the composite function

$$F = f \circ g$$

is differentiable at $x$ and

$$\frac{d}{dx}F(x) = \frac{d}{dx}f(g(x)) = f'(g(x))g'(x)$$

In Liebniz notation: if $y = f(u)$ and $u = g(x)$, then

$$\frac{dy}{dx} = \frac{dy}{du}\frac{du}{dx}.$$

# Example

Determine any inside and outside functions and find the derivative.

(a) $F(x) = \sin^2 x \quad = (\sin x)^2$

$f(u) = u^2$

$g(x) = \sin x$

$F'(x) = f'(g(x))\, g'(x)$

$f'(u) = 2u$

$= 2 \sin x \cos x$

$g'(x) = \cos x$