Sept 9 Math 2253H sec. 05H Fall 2014

Section 2.5: The Chain Rule

Theorem: Suppose g is differentiable at x and f is differentiable at $g(x)$. Then the composite function

$$F = f \circ g$$

is differentiable at x and

$$\frac{d}{dx}F(x) = \frac{d}{dx}f(g(x)) = f'(g(x))g'(x)$$

In Liebniz notation: if $y = f(u)$ and $u = g(x)$, then

$$\frac{dy}{dx} = \frac{dy}{du}\frac{du}{dx}.$$

Identify the parts of the composition, and evaluate the derivative.

(b) $F(x) = \sqrt[3]{x^4 - 5x^2 + 1}$

$= (x^4 - 5x^2 + 1)^{1/3}$

$f(u) = u^{1/3}$

$g(x) = x^4 - 5x^2 + 1$

$f'(u) = \frac{1}{3} u^{-2/3}$

$g'(x) = 4x^3 - 10x$

$F'(x) = f'(g(x))\, g'(x)$

$= \frac{1}{3}(x^4 - 5x^2 + 1)^{-2/3}(4x^3 - 10x)$

$= \dfrac{4x^3 - 10x}{3\sqrt[3]{(x^4 - 5x^2 + 1)^2}}$

(c) $G(\theta) = \cos\left(\sqrt{2\theta}\right)$

$G(\theta) = f(g(h(\theta)))$

$f(v) = \cos(v)$

$g(u) = \sqrt{u} = u^{1/2}$

$h(\theta) = 2\theta$

$f(g(h(\theta))) = f(g(2\theta))$

$= f(\sqrt{2\theta})$

$= \cos(\sqrt{2\theta})$

$G'(\theta) = f'(g(h(\theta))) \cdot \frac{d}{d\theta} g(h(\theta))$

$= f'(g(h(\theta)))\, g'(h(\theta))\, h'(\theta)$

$= -\sin\left(\sqrt{2\theta}\right) \frac{1}{2}(2\theta)^{-1/2} \cdot 2$

$= \dfrac{-\sin(\sqrt{2\theta})}{\sqrt{2\theta}}$

The power rule with the chain rule

If $u = g(x)$ is a differentiable function and n is a real number, then

$$\frac{d}{dx} u^n = n u^{n-1} \frac{du}{dx}.$$

Evaluate: $\frac{d}{dx}\left(\frac{x-1}{2x+2}\right)^7 = 7\left(\frac{x-1}{2x+2}\right)^6 \cdot \frac{d}{dx}\left(\frac{x-1}{2x+2}\right)$

$$= 7\left(\frac{x-1}{2x+2}\right)^6\left(\frac{4}{(2x+2)^2}\right)$$

$$= \frac{28\,(x-1)^6}{(2x+2)^8}$$

Side problem: If $u = \dfrac{x-1}{2x+2}$

$$\frac{du}{dx} = \frac{1(2x+2) - 2(x-1)}{(2x+2)^2} = \frac{2x+2-2x+2}{(2x+2)^2} = \frac{4}{(2x+2)^2}$$

Example

Find the equation of the line tangent to the graph of $y = \cos^4 x$ at the point $(\frac{\pi}{4}, \frac{1}{4})$.

$= (\cos x)^4$

We need the slope:

$$\frac{dy}{dx} = 4(\cos x)^3(-\sin x) = -4\cos^3 x \sin x$$

$$\text{Slope } m = \left.\frac{dy}{dx}\right|_{x=\frac{\pi}{4}} = -4\cos^3\left(\frac{\pi}{4}\right)\sin\left(\frac{\pi}{4}\right) = -4\left(\frac{1}{\sqrt{2}}\right)^3\left(\frac{1}{\sqrt{2}}\right)$$

$$= -4\left(\frac{1}{4}\right) = -1$$

$$y - \frac{1}{4} = -1\left(x - \frac{\pi}{4}\right) \Rightarrow \boxed{y = -x + \frac{\pi}{4} + \frac{1}{4}}$$

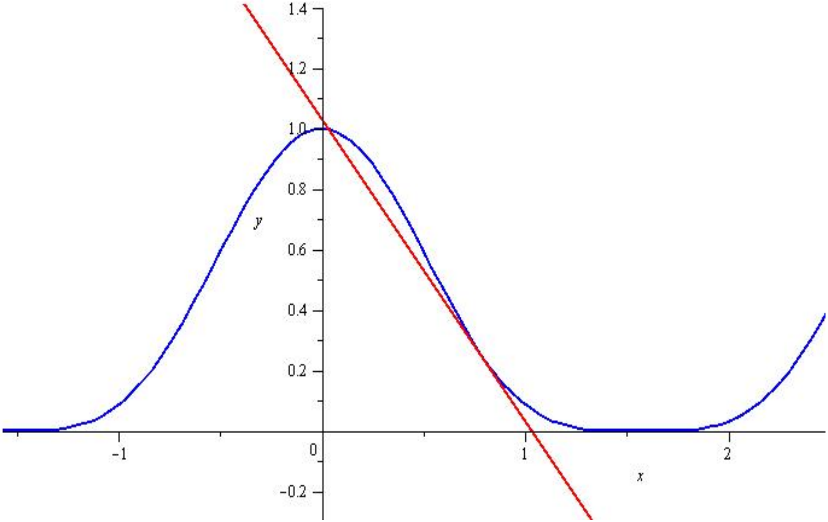


Figure: $y = \cos^4 x$ and the tangent line at $(\frac{\pi}{4}, \frac{1}{4})$.

We may be able to choose between differentiation methods.

Evaluate $\dfrac{d}{dx}\ \dfrac{\sin x}{x^3+2}$ using

(a) The quotient rule:

$$\frac{d}{dx}\frac{\text{Sin}x}{x^3+2} = \frac{\text{Cos}x\,(x^3+2) - 3x^2\,\text{Sin}x}{(x^3+2)^2}$$

(b) writing $\dfrac{\sin x}{x^3+2} = (\sin x)(x^3+2)^{-1}$ and using the chain rule.

$$\frac{d}{dx}\frac{\sin x}{x^3+2} = \frac{d}{dx}\sin x\,(x^3+2)^{-1}$$

$$= \cos x\,(x^3+2)^{-1} + \sin x\left(-1(x^3+2)^{-2}(3x^2)\right)$$

$$= \frac{\cos x}{x^3+2} - \frac{3x^2\sin x}{(x^3+2)^2}$$

$$= \frac{\cos x\,(x^3+2)}{(x^3+2)^2} - \frac{3x^2\sin x}{(x^3+2)^2} = \frac{\cos x\,(x^3+2) - 3x^2\sin x}{(x^3+2)^2}$$

Section 2.7: Applications in Various Sciences

Recall: If x and y are an independent and a dependent variable, respectively, with $y = f(x)$, and x changes from x_1 to x_2, then

$$\text{The change in } x \text{ is} \quad \Delta x = x_2 - x_1.$$

$$\text{The change in } y \text{ is} \quad \Delta y = f(x_2) - (x_1).$$

The average rate of change of y w/r/t x on $[x_1, x_2]$ is

$$\frac{\Delta y}{\Delta x} = \frac{f(x_2) - f(x_1)}{x_2 - x_1}.$$

And the instantaneous rate of change of y w/r/t x at x_1 is

$$\left.\frac{dy}{dx}\right|_{x_1} = \lim_{\Delta x \to 0} \frac{\Delta y}{\Delta x} = \lim_{x_2 \to x_1} \frac{f(x_2) - f(x_1)}{x_2 - x_1}.$$

Position/Velocity/Acceleration

If $s = f(t)$ is the postition of a particle moving along a straight line path (e.g. the x-axis), then

$$\frac{f(t_2) - f(t_1)}{t_2 - t_1} \quad \text{is the average velocity over the time period } t_1 \le t \le t_2.$$

$$v(t_1) = s'(t_1) \quad \text{is the velocity of the particle at time } t = t_1.$$

$$a(t_1) = v'(t_1) = s''(t_1) \quad \text{is the velocity of the particle at time } t = t_1.$$

Example

A particle moves along the x-axis so that its position s in feet at time t in seconds is

$$s = t^3 - 9t^2 + 24t + 5$$

(a) Find the posititions of the particle at times $t = 0$ and $t = 2$.

$s(0) = 5$ ft

$s(2) = 2^3 - 9(2^2) + 24(2) + 5$

$= 8 - 36 + 48 + 5 = 25$ ft

$s(0) = 5$ ft $\quad s(2) = 25$ ft

(b) Find the average velocity of the particle for $0 \leq t \leq 2$.

avg. vel. $= \dfrac{s(2) - s(0)}{2 - 0} = \dfrac{25\text{ ft} - 5\text{ ft}}{2\text{ sec} - 0\text{ sec}} = 10\ \dfrac{\text{ft}}{\text{sec}}$

$s = t^3 - 9t^2 + 24t + 5$

(c) Determine the velocity $v(t)$ of the particle.

$v(t) = s'(t) = 3t^2 - 18t + 24$

(d) Determine the acceleration $a(t)$ of the particle.

$a(t) = v'(t) = s''(t) = 6t - 18$

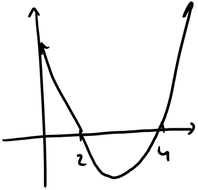
$s = t^3 - 9t^2 + 24t + 5$

(e) When is the particle moving to the right?

The particle moves to the right when $v(t) > 0$.

$v(t) = 3t^2 - 18t + 24 = 3(t^2 - 6t + 8) = 3(t-2)(t-4)$

$v(t) > 0$ if $0 \le t < 2$ or $t > 4$



The particle moves to the right when $0 \le t < 2$ and when $t > 4$.

(f) When is the particle at rest?

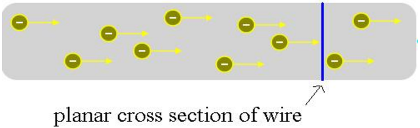
"At rest" means change in position is zero.

i.e. $V(t) = 0$

$V(t) = 3(t-4)(t-2) \Rightarrow V(t) = 0$ if $t = 2$ or $t = 4$.

Electro-Magnetics

When electrons move through a wire, a change in electric charge occurs. Rate of change of charge is current.



If the charge on the cross section is Q_1 at time t_1 and Q_2 at time t_2, then

$$\text{average current over this interval} \quad = \frac{\Delta Q}{\Delta t} = \frac{Q_2 - Q_1}{t_2 - t_1}.$$

The quantities current $I(t)$ and charge $Q(t)$ are related by $\quad I(t) = \dfrac{dQ}{dt}.$

Example

Ohm's law states that the voltage drop across a resistor is proportional to the current. The constant of proportionality R is called the *resistance*. Mathematically, the potential difference V (in volts), current I (in amperes), and resistance R (in ohms) satisfy

$$V = IR.$$

Suppose the charge at time t is known to be $Q(t) = q_0 \sin(\omega t)$ where q_0 and ω are constant. If the resistance is $R = 100$ ohms, express V as a function of t in terms of q_0 and ω.

$V = IR$ and $I = \frac{dQ}{dt}$

$$Q(t) = q_0 \sin(\omega t)$$

$$I = \frac{dQ}{dt} = q_0 \cos(\omega t) \cdot \omega$$

$$I = q_0 \omega \cos(\omega t)$$

$$\boxed{V = RI = 100 q_0 \omega \cos(\omega t)}$$