January 10 Math 3260 sec. 55 Spring 2020

Section 1.1: Systems of Linear Equations

We saw an example of solving the system

$$\begin{array}{rcrcrcr} x_1 & + & 2x_2 & - & x_3 & = & -4 \\ 2x_1 & & & + & x_3 & = & 7 \\ x_1 & + & x_2 & + & x_3 & = & 6 \end{array}$$

We obtained the solution by doing operations to reduce it to the **equivalent system**

$$\begin{array}{rcrcrcr} x_1 & + & 2x_2 & - & x_3 & = & -4 \\ & & x_2 & - & 2x_3 & = & -10 \\ & & & & x_3 & = & 5 \end{array}$$

Those Operations for Solving a System...

- swap the order of any two equations **(swap)**,

$$E_i \leftrightarrow E_j$$

- multiply an equation by any nonzero constant **(scale)**,

$$kE_i \to E_i \quad \text{and}$$

- replace an equation with the sum of itself and a nonzero multiple of any other equation **(replace).**

$$kE_i + E_j \to E_j$$

Matrices

Definition: A matrix is a rectangular array of numbers. It's **size** (a.k.a. dimension/order) is $m \times n$ (read "m by n") where m is the number of rows and n is the number of columns the matrix has.

Examples:

4 columns

3 rows
$$\begin{bmatrix} 2 & 0 & -1 & 3 \\ 1 & 1 & 13 & -4 \\ 12 & -3 & 2 & -2 \end{bmatrix},$$

3×4

2 columns

3 rows
$$\begin{bmatrix} 2 & 0 \\ 4 & 4 \\ 3 & -5 \end{bmatrix}$$

3×2

Linear System & Matrices

Given any linear system of equations, we can associate two matrices with the system. These are the **coefficient** matrix and the **augmented** matrix.

Example: $$\begin{array}{rcrcrcr} x_1 & + & 2x_2 & - & x_3 & = & -4 \\ 2x_1 & & & + & x_3 & = & 7 \\ x_1 & + & x_2 & + & x_3 & = & 6 \end{array}$$

Before we start to set up these matrices, we write our system in the form shown above. Note that all variables are on the left side, and like variables have the same order in each equation (they are aligned vertically).

Linear System: Coefficient Matrix

The **coefficient** matrix has one row for each equation and one column for each variable. The entries are the coefficients of the variables in our system.

3 equations

$m = 3$

Example:
$$\begin{array}{rcrcrcr} x_1 & + & 2x_2 & - & x_3 & = & -4 \\ 2x_1 & & & + & x_3 & = & 7 \\ x_1 & + & x_2 & + & x_3 & = & 6 \end{array}$$

3 variables

$n = 3$

$$\begin{bmatrix} 1 & 2 & -1 \\ 2 & 0 & 1 \\ 1 & 1 & 1 \end{bmatrix}$$

Linear System: Augmented Matrix

The **augmented** matrix has one row for each equation, one column for each variable, and one extra, right most column. The entries in the first columns match the coefficient matrix, and the right most column has the numbers from the right hand side of each equation.

Example:
$$\begin{array}{rcrcrcr} x_1 & + & 2x_2 & - & x_3 & = & -4 \\ 2x_1 & & & + & x_3 & = & 7 \\ x_1 & + & x_2 & + & x_3 & = & 6 \end{array}$$

3 eqns
$m=3$
3 variables
$n=3+1=4$

$$\begin{bmatrix} 1 & 2 & -1 & -4 \\ 2 & 0 & 1 & 7 \\ 1 & 1 & 1 & 6 \end{bmatrix}$$

A key here is *structure*!

Consider the following augmented matrix. Determine if the associated system is consistent or inconsistent. If it is consistent, determine the solution set.

(a) $\left[\begin{array}{rrrr} 1 & 0 & 0 & 3 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & -2 \end{array}\right]$

$x_1 = 3$

$x_2 = 1$

$x_3 = -2$

The system is consistent w/ solution set $\{(3, 1, -2)\}$.

(b) $\begin{bmatrix} 1 & 2 & 0 & 3 \\ 0 & 1 & -1 & 4 \\ 0 & 0 & 0 & 3 \end{bmatrix}$

$$x_1 + 2x_2 = 3$$
$$x_2 - x_3 = 4$$
$$0x_1 + 0x_2 + 0x_3 = 3$$

$0 = 3$ always false!

The system is inconsistent.

The solution set is empty ϕ

(c) $\begin{bmatrix} 1 & 1 & -1 & 0 \\ 0 & 1 & 1 & 4 \\ 0 & 0 & 0 & 0 \end{bmatrix}$

$$x_1 + x_2 - x_3 = 0$$
$$x_2 + x_3 = 4$$
$$0x_1 + 0x_2 + 0x_3 = 0$$
$$0 = 0$$

always true

The system is consistent.

The 1st 2 equations give

$$x_1 = -x_2 + x_3$$
$$x_2 = 4 - x_3$$

$\Rightarrow$ $x_1 = -(4 - x_3) + x_3$

$= -4 + 2x_3$

(x_1, x_2, x_3) is a solution if

$$x_1 = -4 + 2x_3$$

$x_2 = 4 - x_3$

for any real x_3

Triangular Shape

Recall our two equivalent systems. Let's look at their augmented matrices:

$$\begin{array}{rcrcrcr} x_1 & + & 2x_2 & - & x_3 & = & -4\\ 2x_1 & & & + & x_3 & = & 7\\ x_1 & + & x_2 & + & x_3 & = & 6 \end{array} \qquad \begin{bmatrix} 1 & 2 & -1 & -4\\ 2 & 0 & 1 & 7\\ 1 & 1 & 1 & 6 \end{bmatrix}$$

No particular special structure.

$$\begin{array}{rcrcrcr} x_1 & + & 2x_2 & - & x_3 & = & -4\\ & & x_2 & - & 2x_3 & = & -10\\ & & & & x_3 & = & 5 \end{array} \qquad \begin{bmatrix} 1 & 2 & -1 & -4\\ 0 & 1 & -2 & -10\\ 0 & 0 & 1 & 5 \end{bmatrix}$$

Has a sort of triangular structure with 1's down the diagonal.

Section 1.2: Row Reduction and Echelon Forms

We defined the following **Elementary Row Operations** that can be performed on a matrix.

i Interchange any two rows (**row swap**).

ii Multiply a row by any nonzero constant (**scaling**).

iii Replace a row with the sum of itself and a multiple of another row (**replacement**).

Definition: Two matrices are said to be **row equivalent** if they can be transformed into one another by a sequence of elementary row operations.

Theorem on Row Equivalent Matrices

Theorem: If the augmented matrices of two linear systems are row equivalent, then the systems are equivalent.

Recall that two linear systems of equations are called *equivalent* if they have the same solution set.

Why is Row Equivalence Interesting?

The following matrices are row equivalent.

$$\begin{bmatrix} 1 & -1 & -1 & 7 \\ 1 & 2 & 2 & 1 \\ \frac{1}{2} & \frac{7}{2} & 1 & 3 \end{bmatrix}$$

$$\begin{bmatrix} 1 & 0 & 0 & 3 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & -2 \end{bmatrix}$$

Echelon Forms

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Definition: A matrix is in **echelon form** (a.k.a. **row echelon form**) if the following properties hold

i. Any row of all zeros are at the bottom.
ii. The first nonzero number (called the *leading entry*) in a row is to the right of the first nonzero number in all rows above it.
iii. All entries below a leading entry are zeros.

Is

$$\begin{bmatrix} 2 & 1 & 3 \\ 0 & -1 & 1 \\ 0 & 0 & 7 \end{bmatrix}$$

Is Not

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 2 & -3 \\ 0 & 1 & 0 \\ 0 & 0 & 4 \end{bmatrix}$$

violates ii and iii

Reduced Echelon Form

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Definition: A matrix is in **reduced echelon form** (a.k.a. **reduced row echelon form**) if it is in echelon form and the following additional properties hold

iv The leading entry of each row is 1 (called a *leading* 1), and

v each leading 1 is the only nonzero entry in its column.

Is

$$\begin{bmatrix} 1 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}$$

Is Not

not zero

$$\begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}$$