January 11 Math 2306 sec. 53 Spring 2019

Section 1: Concepts and Terminology

We define solutions of the nth order ODE $F(x, y, y', \ldots, y^{(n)}) = 0$ (*)

Definition: A function ϕ defined on an interval[1] I and possessing at least n continuous derivatives on I is a **solution** of (*) on I if upon substitution (i.e. setting $y = \phi(x)$) the equation reduces to an identity.

Definition: An **implicit solution** of (*) is a relation $G(x, y) = 0$ provided there exists at least one function $y = \phi$ that satisfies both the differential equation (*) and this relation.

[1] The interval is called the *domain of the solution* or the *interval of definition*.

An Implicit Solution

Verify that the relation(left) defines and implicit solution of the differential equation (right).

$$y^2 - 2x^2y = 1, \qquad \frac{dy}{dx} = \frac{2xy}{y - x^2}$$

We'll assume that $y^2 - 2x^2y = 1$ is true, and show that the differential equation must therefore be true.

We start with $y^2 - 2x^2y = 1$ and find $\frac{dy}{dx}$ using implicit differentiation.

$$2y\frac{dy}{dx} - 2\left(2xy + x^2\frac{dy}{dx}\right) = 0$$

It may not be possible to clearly identify the domain of definition of an implicit solution.

Solve for $\frac{dy}{dx}$

$$y\frac{dy}{dx} - 2xy - x^2\frac{dy}{dx} = 0$$

$$(y-x^2)\frac{dy}{dx} = 2xy$$

$$\frac{dy}{dx} = \frac{2xy}{y-x^2}$$

So $y^2 - 2x^2y = 1$ does imply that y satisfies the ODE.

Function vs Solution

The interval of defintion has to be an interval.

Example: Consider the ODE $\quad y' = -y^2. \quad$ A solution is $\; y = \dfrac{1}{x}.$

The interval of defintion can be

$$(-\infty, 0) \quad \text{or} \quad (0, \infty)$$

or any interval that doesn't contain the origin.

But it CAN't be $(-\infty, 0) \cup (0, \infty)$ because this isn't an interval!

Depending on other available information, we often assume I is the largest (or one of the largest) possible intervals.

Function vs Solution

The graph of an ODE solution[2] will not have disconnected pieces.

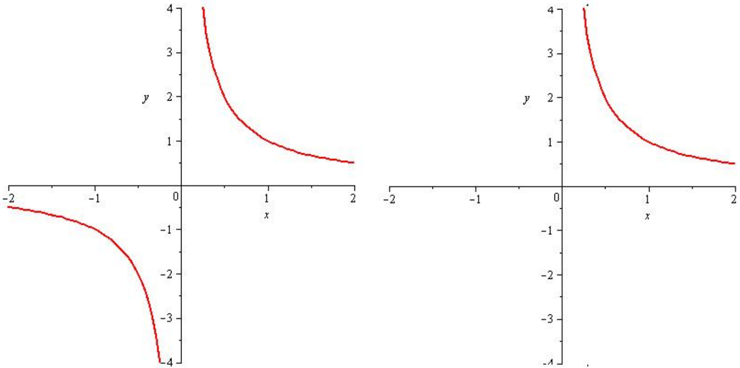


Figure: Left: Plot of $f(x) = \frac{1}{x}$ as a **function**. Right: Plot of $f(x) = \frac{1}{x}$ as a possible **solution** of an ODE.

[2]This is known as a *classical solution*.

Unspecified Constants in a Function

Show that for any choice of constants c_1 and c_2, $y = c_1 x + \frac{c_2}{x}$ is a solution of the differential equation

$$x^2 y'' + xy' - y = 0$$

We'll show that the ODE is true when we substitute without putting any conditions on the numbers c_1 and c_2.

$$y = c_1 x + c_2 x^{-1}$$

$$y' = c_1 - c_2 x^{-2}$$

$$y'' = 0 + 2c_2 x^{-3}$$

$$x^2 y'' + x y' - y =$$

$$x^2\left(2c_2 x^{-3}\right) + x\left(c_1 - c_2 x^{-2}\right) - \left(c_1 x + c_2 x^{-1}\right) =$$

$$2c_2 x^{-1} + c_1 x - c_2 x^{-1} - c_1 x - c_2 x^{-1} =$$

$$x^{-1}\left(2c_2 - c_2 - c_2\right) + x\left(c_1 - c_1\right) =$$

$$x^{-1}(0) + x(0) = 0$$

$$0 = 0$$

Note this holds for any pair of numbers c_1, c_2.

Some Terms

- A **parameter** is an unspecified constant such as c_1 and c_2 in the last example.

- A **family of solutions** is a collection of solution functions that only differ by a parameter.
- An ***n*-parameter family of solutions** is one containing n parameters (e.g. $c_1x + \frac{c_2}{x}$ is a 2 parameter family).

- A **particular solution** is one with no arbitrary constants in it.

- The **trivial solution** is the simple constant function $y = 0$.

- An **integral curve** is the graph of one solution (perhaps from a family).

Section 2: Initial Value Problems

An initial value problem consists of an ODE with a certain type of additional conditions.

Solve the equation [3]

$$\frac{d^n y}{dx^n} = f(x, y, y', \dots, y^{(n-1)}) \tag{1}$$

subject to the *initial conditions*

$$y(x_0) = y_0, \quad y'(x_0) = y_1, \quad \dots, y^{(n-1)}(x_0) = y_{n-1}. \tag{2}$$

The problem (1)–(2) is called an *initial value problem* (IVP).

[3] on some interval I containing x_0.

IVPs

First order case:

1st order

one condition

$$\frac{dy}{dx} = f(x,y), \quad y(x_0) = y_0$$

The ODE part tells us what the slope of the curve is at each point. The initial condition tells us the graph of y passes through (x_0, y_0).

Second order case:

2nd order

two conditions

$$\frac{d^2y}{dx^2} = f(x,y,y'), \quad y(x_0) = y_0, \quad y'(x_0) = y_1$$

If y is the position of a particle at time x, the ODE specifies its acceleration.

y_0: its initial position, y_1 = its initial velocity

Example

Given that $y = c_1 x + \frac{c_2}{x}$ is a 2-parameter family of solutions of $x^2 y'' + xy' - y = 0$, solve the IVP

$$x^2 y'' + xy' - y = 0, \quad y(1) = 1, \quad y'(1) = 3$$

We already know that solutions to the ODE look like $y = c_1 x + \frac{c_2}{x}$. We need to find values of c_1, c_2 so that $y(1) = 1$ and $y'(1) = 3$.

$$y(x) = c_1 x + \frac{c_2}{x} \qquad y(1) = c_1(1) + \frac{c_2}{1} = 1$$

$$y'(x) = c_1 - \frac{c_2}{x^2} \qquad y'(1) = c_1 - \frac{c_2}{1^2} = 3$$

$$c_1 + c_2 = 1$$

$$c_1 - c_2 = 3$$

add the equations

$$2c_1 = 4 \implies c_1 = 2$$

$$c_1 + c_2 = 1 \implies c_2 = 1 - c_1 = 1 - 2 = -1$$

The solution to the IVP is

$$\boxed{y = 2x - \frac{1}{x}.}$$

Example

Part 1

Show that for any constant c the relation $x^2 + y^2 = c$ is an implicit solution of the ODE

$$\frac{dy}{dx} = -\frac{x}{y}$$

From $x^2+y^2=C$, $\quad 2x + 2y\frac{dy}{dx} = 0$

$$2y\frac{dy}{dx} = -2x$$

$$\frac{dy}{dx} = \frac{-2x}{2y} = \frac{-x}{y} \quad (\text{for } y \neq 0)$$

Example

Part 2

Use the preceding results to find an explicit solution of the IVP

$$\frac{dy}{dx} = -\frac{x}{y}, \quad y(0) = -2$$

We know that $x^2+y^2=C$ for some number C.

Setting $y=-2$ when $x=0$

$$0^2+(-2)^2=C \implies C=4$$

So $x^2+y^2=4$ is an implicit solution of the IVP.

To find an explicit solution solve for y.

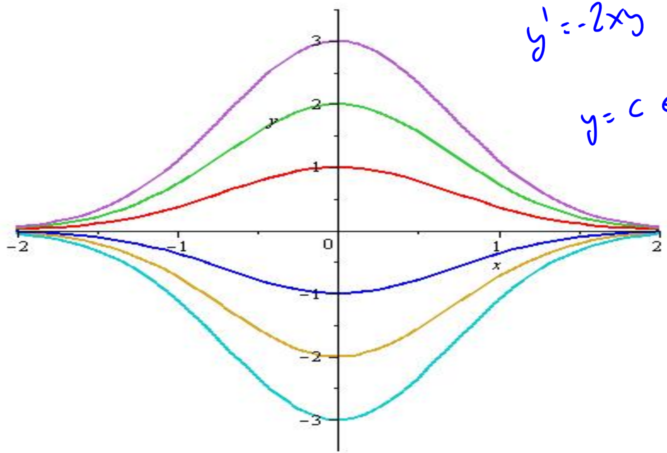
$$y^2 = 4 - x^2 \quad \Rightarrow \quad y = \sqrt{4-x^2} \quad \text{or}$$

$$y = -\sqrt{4-x^2}$$

Since $y(0) = -2$, the explicit solution to the IVP is

$$y = -\sqrt{4-x^2}$$

Graphical Interpretation



$y' = -2xy$

$y = c\, e^{-x^2}$

Figure: Each curve solves $y' + 2xy = 0$, $y(0) = y_0$. Each colored curve corresponds to a different value of y_0