## January 11 Math 2306 sec. 57 Spring 2018

## Section 1: Concepts and Terminology

Recall that we defined a Differential Equation as an equation containing the derivative(s) of one or more dependent variables, with respect to one or more indendent variables.

Solving a differential equation refers to determining the dependent variable(s)-as function(s).

Independent Variable: will appear as one that derivatives are taken with respect to.

Dependent Variable: will appear as one that derivatives are taken of.

## Classifications

We defined the order and linearity
Order: The order of a differential equation is the same as the highest order derivative appearing anywhere in the equation.

Linearity: An $n^{\text {th }}$ order differential equation is said to be linear if it can be written in the form

$$
a_{n}(x) \frac{d^{n} y}{d x^{n}}+a_{n-1}(x) \frac{d^{n-1} y}{d x^{n-1}}+\cdots+a_{1}(x) \frac{d y}{d x}+a_{0}(x) y=g(x) .
$$

Note that each of the coefficients $a_{0}, \ldots, a_{n}$ and the right hand side $g$ may depend on the independent variable but not on the dependent variable or any of its derivatives.

## Solution of $F\left(x, y, y^{\prime}, \ldots, y^{(n)}\right)=0\left(^{*}\right)$

$$
\begin{aligned}
& \text { recall, } \uparrow \text { this is ow generic } n^{\text {th }} \text { orden } \\
& \text { ODE }
\end{aligned}
$$

Definition: A function $\phi$ defined on an interval ${ }^{1} I$ and possessing at least $n$ continuous derivatives on $I$ is a solution of (*) on $I$ if upon substitution (i.e. setting $y=\phi(x)$ ) the equation reduces to an identity.

Definition: An implicit solution of $\left(^{*}\right)$ is a relation $G(x, y)=0$ provided there exists at least one function $y=\phi$ that satisfies both the differential equation (*) and this relation.

[^0]Examples:
Verify that the given function is an solution of the ODE on the indicated interval.

$$
\phi(t)=t^{-4}-3, \quad I=(0, \infty), \quad t^{2} \frac{d^{2} y}{d t^{2}}+2 t \frac{d y}{d t}-12 y=36
$$

$\Phi$ has derivatives of all orders for $\alpha>0$.

$$
\begin{aligned}
\phi^{\prime}=-4 t^{5}, \phi^{\prime \prime} & =20 t^{-6} \quad \text { so setting } y=\phi \\
t^{2} y^{\prime \prime}+2 t y^{\prime}-12 y & =t^{2}\left(20 t^{-6}\right)+2 t\left(-4 t^{-5}\right)-12\left(t^{-4}-3\right) \\
& =20 t^{-4}-8 t^{-4}-12 t^{-4}+36 \\
& =(20-8-12) t^{-4}+36 \quad=36 \\
0^{\prime \prime} \quad 36 & =36
\end{aligned}
$$

So $\phi$ is a solution.

Verify that the relation(left) defines and implicit solution of the differential equation (right).

$$
y^{2}-2 x^{2} y=1, \quad \frac{d y}{d x}=\frac{2 x y}{y-x^{2}}
$$

will use implicit diffentiction to show that If $y$ satisfies the relation, it also satisfies the $O D E$.

$$
\begin{aligned}
& y^{2}-2 x^{2} y=1 \Rightarrow 2 y \frac{d y}{d x}-\left(4 x y+2 x^{2} \frac{d y}{d x}\right)=0 \\
& 2 y \frac{d y}{d x}-4 x y-2 x^{2} \frac{d y}{d x}=0 \quad \begin{array}{l}
\text { Div. by } 2 \\
\text { collect } y^{\prime}
\end{array} \\
&\left(y-x^{2}\right) \frac{d y}{d x}-2 x y=0 \\
& \Rightarrow\left(y-x^{2}\right) \frac{d y}{d x}= \\
& 2 x y \text { finally } \frac{d y}{d x}=\frac{2 x y}{y-x^{2}} \text { as expected! }
\end{aligned}
$$

It may not be possible to clearly identify the domain of definition of an implicit solution. Here, we se $y-x^{2} \neq 0$.

## Function vs Solution

## The interval of defintion has to be an interval.

Consider $y^{\prime}=-y^{2}$. Clearly $y=\frac{1}{x}$ solves the DE. The interval of defintion can be $(-\infty, 0)$, or $(0, \infty)$-or any interval that doesn't contain the origin. But it can't be $(-\infty, 0) \cup(0, \infty)$ because this isn't an interva!!

Often, we'll take / to be the largest, or one of the largest, possible intervasl. It may depend on other information.



Figure: Left: Plot of $f(x)=\frac{1}{x}$ as a function. Right: Plot of $f(x)=\frac{1}{x}$ as a possible solution of an ODE.

Show that for any choice of constants $c_{1}$ and $c_{2}, y=c_{1} x+\frac{c_{2}}{x}$ is a solution of the differential equation

$$
x^{2} y^{\prime \prime}+x y^{\prime}-y=0
$$

Finding $y$ and $y^{\prime \prime}$

$$
\begin{aligned}
& y^{\prime}=c_{1} 1+c_{2}\left(-x^{-2}\right)=c_{1}-c_{2} x^{-2} \\
& y^{\prime \prime}=-c_{2}\left(-2 x^{-3}\right)=2 c_{2} x^{-3}
\end{aligned}
$$

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$$
\begin{aligned}
& x^{2} y^{\prime \prime}+x y^{\prime}-y=x^{2}\left(2 c_{2} x^{-3}\right)+x\left(c_{1}-c_{2} x^{-2}\right)-\left(c_{1} x+c_{2} x^{-1}\right) \\
&=2 c_{2} x^{-1}+c_{1} x-c_{2} x^{-1}-c_{1} x-c_{2} x^{-1} \\
&=x^{-1}\left(2 c_{2}-c_{2}-c_{2}\right)+x\left(c_{1}-c_{1}\right) \\
& 0_{0}^{\prime \prime}
\end{aligned}
$$

$$
=0
$$

Hence $b$ is a solution. Note that the $C_{1}$ and $C_{2}$ cancel no matter what value then take.

* We do require they are constant. If not, our derivatives ane all wrong!


## Some Terms

- A parameter is an unspecified constant such as $c_{1}$ and $c_{2}$ in the last example.
- A family of solutions is a collection of solution functions that only differ by a parameter.
- An $n$-parameter family of solutions is one containing $n$ parameters (e.g. $c_{1} x+\frac{c_{2}}{x}$ is a 2 parameter family).
- A particular solution is one with no arbitrary constants in it.
- The trivial solution is the simple constant function $y=0$.
- An integral curve is the graph of one solution (perhaps from a family).


[^0]:    ${ }^{1}$ The interval is called the domain of the solution or the interval of definition.

