January 11 Math 2306 sec. 60 Spring 2018

Section 1: Concepts and Terminology

Recall that we defined a **Differential Equation** as an equation containing the derivative(s) of one or more dependent variables, with respect to one or more indendent variables.

Solving a differential equation refers to determining the dependent variable(s)—as function(s).

Independent Variable: will appear as one that derivatives are taken with respect to.

Dependent Variable: will appear as one that derivatives are taken of.

Classifications

We defined the order and linearity

Order: The order of a differential equation is the same as the highest order derivative appearing anywhere in the equation.

Linearity: An n^{th} order differential equation is said to be **linear** if it can be written in the form

$$a_n(x)\frac{d^n y}{dx^n} + a_{n-1}(x)\frac{d^{n-1} y}{dx^{n-1}} + \cdots + a_1(x)\frac{dy}{dx} + a_0(x)y = g(x).$$

Note that each of the coefficients $a_0, \ldots, a_n$ and the right hand side g may depend on the independent variable but not on the dependent variable or any of its derivatives.

Solution of $F(x, y, y', \ldots, y^{(n)}) = 0$ (*)

recall, ↑ this is our generic nth order ODE

Definition: A function ϕ defined on an interval[1] I and possessing at least n continuous derivatives on I is a **solution** of (*) on I if upon substitution (i.e. setting $y = \phi(x)$) the equation reduces to an identity.

Definition: An **implicit solution** of (*) is a relation $G(x, y) = 0$ provided there exists at least one function $y = \phi$ that satisfies both the differential equation (*) and this relation.

[1] The interval is called the *domain of the solution* or the *interval of definition*.

Examples:

Verify that the given function is an solution of the ODE on the indicated interval.

$$\phi(t) = t^{-4} - 3, \quad I = (0, \infty), \quad t^2 \frac{d^2y}{dt^2} + 2t\frac{dy}{dt} - 12y = 36$$

ϕ has derivatives of all orders for $x > 0$.

$\phi' = -4t^{-5}$, $\phi'' = 20t^{-6}$ so setting $y = \phi$

$$t^2y'' + 2ty' - 12y = t^2(20t^{-6}) + 2t(-4t^{-5}) - 12(t^{-4} - 3)$$

$$= 20t^{-4} - 8t^{-4} - 12t^{-4} + 36$$

$$= (\underbrace{20 - 8 - 12}_{0})t^{-4} + 36 \qquad = 36$$

$36 = 36$!

So ϕ is a solution.

Verify that the relation(left) defines and implicit solution of the differential equation (right).

$$y^2 - 2x^2y = 1, \qquad \frac{dy}{dx} = \frac{2xy}{y - x^2}$$

We'll use implicit differentiation to show that if y satisfies the relation, it also satisfies the ODE.

$$y^2 - 2x^2y = 1 \implies 2y\frac{dy}{dx} - \left(4xy + 2x^2\frac{dy}{dx}\right) = 0$$

$$2y\frac{dy}{dx} - 4xy - 2x^2\frac{dy}{dx} = 0$$

Div. by 2
collect y'

$$(y - x^2)\frac{dy}{dx} - 2xy = 0$$

$$\implies (y-x^2)\frac{dy}{dx} = 2xy \quad \text{finally} \quad \frac{dy}{dx} = \frac{2xy}{y-x^2}$$

as expected!

It may not be possible to clearly identify the domain of definition of an implicit solution. Here, we see $y - x^2 \neq 0$.

Function vs Solution

The interval of defintion has to be an interval.

Consider $y' = -y^2$. Clearly $y = \frac{1}{x}$ solves the DE. The interval of defintion can be $(-\infty, 0)$, or $(0, \infty)$—or any interval that doesn't contain the origin. But it can't be $(-\infty, 0) \cup (0, \infty)$ because this isn't an interval!

Often, we'll take I to be the largest, or one of the largest, possible intervasl. It may depend on other information.

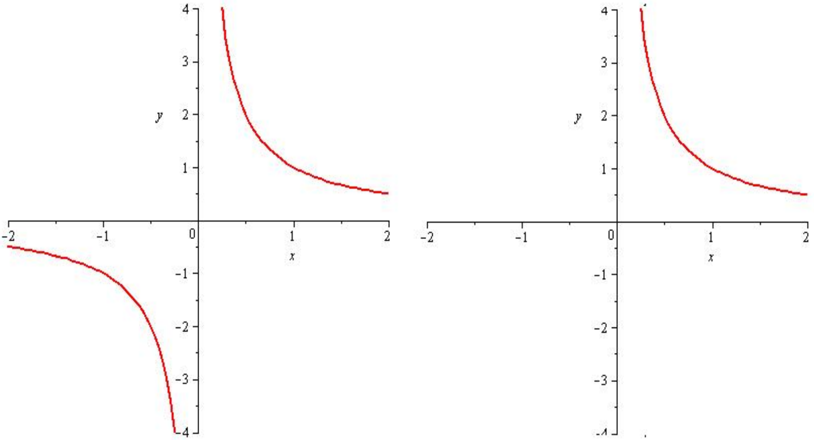

Figure: Left: Plot of $f(x) = \frac{1}{x}$ as a **function**. Right: Plot of $f(x) = \frac{1}{x}$ as a possible **solution** of an ODE.

Show that for any choice of constants c_1 and c_2, $y = c_1 x + \frac{c_2}{x}$ is a solution of the differential equation

$$x^2 y'' + xy' - y = 0$$

Finding y' and y''

$y' = c_1 \cdot 1 + c_2(-x^{-2}) = c_1 - c_2 x^{-2}$

$y'' = -c_2(-2x^{-3}) = 2c_2 x^{-3}$

Substitute

$x^2 y'' + xy' - y = x^2(2c_2 x^{-3}) + x(c_1 - c_2 x^{-2}) - (c_1 x + c_2 x^{-1})$

$= 2c_2 x^{-1} + c_1 x - c_2 x^{-1} - c_1 x - c_2 x^{-1}$

$= x^{-1}(2c_2 - c_2 - c_2) + x(c_1 - c_1)$

"0" "0"

$= 0$

Hence y is a solution. Note that the c_1 and c_2 cancel no matter what value they take.

* We do require they are constant.
If not, our derivatives are all wrong!

Some Terms

- A **parameter** is an unspecified constant such as c_1 and c_2 in the last example.
- A **family of solutions** is a collection of solution functions that only differ by a parameter.
- An ***n*-parameter family of solutions** is one containing n parameters (e.g. $c_1x + \frac{c_2}{x}$ is a 2 parameter family).
- A **particular solution** is one with no arbitrary constants in it.
- The **trivial solution** is the simple constant function $y = 0$.
- An **integral curve** is the graph of one solution (perhaps from a family).