January 11 Math 3260 sec. 56 Spring 2018

Section 1.1: Systems of Linear Equations

We had the following theorem on linear systems:

Theorem: A linear system of equations has exactly one of the following:

i No solution, or

ii Exactly one solution, or

iii Infinitely many solutions.

We use the terms:

Consistent if it has at least one solution, and
Inconsistent if it has no solution.

Also recall...

Given a linear system, we can equate two matrices with it. The coefficient matrix, and the augmented matrix. For example

$$\text{Example:}\quad \begin{array}{rcrcrcr} x_1 & + & 2x_2 & - & x_3 & = & -4 \\ 2x_1 & & & + & x_3 & = & 7 \\ x_1 & + & x_2 & + & x_3 & = & 6 \end{array}$$

has coefficient matrix

$$\begin{bmatrix} 1 & 2 & -1 \\ 2 & 0 & 1 \\ 1 & 1 & 1 \end{bmatrix}$$

and augmented matrix

$$\begin{bmatrix} 1 & 2 & -1 & -4 \\ 2 & 0 & 1 & 7 \\ 1 & 1 & 1 & 6 \end{bmatrix}$$

Legitimate Operations for Solving a System

We can perform three basic operation **without** changing the solution set of a system. These are

- swap the order of any two equations **(swap)**,
- multiply an equation by any nonzero constant **(scale)**, and
- replace an equation with the sum of itself and a nonzero multiple of any other equation **(replace)**.

We said that two systems with the same solution set are called **equivalent**.

Some Operation Notation

Notation

- Swap equations i and j:
$$Ei \leftrightarrow Ej$$
- Scale equation i by k:
$$kEi \rightarrow Ei$$
- Replace equation j with the sum of itself and k times equation i:
$$kEi + Ej \rightarrow Ej$$

Solve the following system of equations by *elimination*. Keep tabs on the augmented matrix at each step.

$$\begin{array}{rcrcrcr} x_1 & + & 2x_2 & - & x_3 & = & -4 \\ 2x_1 & & & + & x_3 & = & 7 \\ x_1 & + & x_2 & + & x_3 & = & 6 \end{array}$$

We'll use the operations to eliminate some variables from some eqns.

* Caveat: I make no claims that this is the only "right" way to do this.

Lets get rid of x_1 in the last two equations

Do $-2E_1 + E_2 \to E_2$

$$\begin{array}{rcrcrcr} x_1 & + & 2x_2 & - & x_3 & = & -4 \\ & & -4x_2 & + & 3x_3 & = & 15 \\ x_1 & + & x_2 & + & x_3 & = & 6 \end{array}$$

$$\begin{bmatrix} 1 & 2 & -1 & -4 \\ 0 & -4 & 3 & 15 \\ 1 & 1 & 1 & 6 \end{bmatrix}$$

Do $-E_1 + E_3 \to E_3$

$$\begin{aligned} x_1 + 2x_2 - x_3 &= -4 \\ -4x_2 + 3x_3 &= 15 \\ -x_2 + 2x_3 &= 10 \end{aligned}$$

$$\begin{bmatrix} 1 & 2 & -1 & -4 \\ 0 & -4 & 3 & 15 \\ 0 & -1 & 2 & 10 \end{bmatrix}$$

Lets swap $E_2 \leftrightarrow E_3$

$$\begin{aligned} x_1 + 2x_2 - x_3 &= -4 \\ -x_2 + 2x_3 &= 10 \\ -4x_2 + 3x_3 &= 15 \end{aligned}$$

$$\begin{bmatrix} 1 & 2 & -1 & -4 \\ 0 & -1 & 2 & 10 \\ 0 & -4 & 3 & 15 \end{bmatrix}$$

Do $-4 . E_2 + E_3 \to E_3$

$$\begin{aligned} x_1 + 2x_2 - x_3 &= -4 \\ -x_2 + 2x_3 &= 10 \\ -5x_3 &= -25 \end{aligned}$$

$$\begin{bmatrix} 1 & 2 & -1 & -4 \\ 0 & -1 & 2 & 10 \\ 0 & 0 & -5 & -25 \end{bmatrix}$$

Finally, lets scale

$-E_2 \to E_2$ and $\frac{-1}{3}E_3 \to E_3$

$$\begin{aligned} x_1 + 2x_2 - x_3 &= -4 \\ x_2 - 2x_3 &= -10 \\ x_3 &= 5 \end{aligned}$$

$$\begin{bmatrix} 1 & 2 & -1 & -4 \\ 0 & 1 & -2 & -10 \\ 0 & 0 & 1 & 5 \end{bmatrix}$$

Note the triangular structure on the nonzero entries!

Now x_3 is known.

with some substitution, we get x_2 and then x_1!

From Eqn 2

$$x_2 = -10 + 2x_3 = -10 + 2 \cdot 5 = 0$$

From Eqn 1

$$x_1 = -4 - 2x_2 + x_3 = -4 - 0 + 5 = 1$$

The solution is

$$x_1 = 1, \quad x_2 = 0, \quad x_3 = 5.$$

We could write this as $(1, 0, 5)$.

Example

We used the legitimate equation operations to solve the system by eliminating some variables and then substituting. Our augmented matrix after so many steps looked like

$$\begin{bmatrix} 1 & 2 & -1 & -4 \\ 0 & 1 & 2 & -10 \\ 0 & 0 & 1 & 5 \end{bmatrix}$$

The operations on the equations correspond to changes to the rows of the matrix. This suggests that we can do operations on the rows of matrix to get desired (legitimate) changes to the equations.

Elementary Row Operations

If any sequence of the following operations are performed on a matrix, the resulting matrix is **row equivalent**.

i Replace a row with the sum of itself and a multiple of another row (**replacement**).

ii Interchange any two rows (**row swap**).

iii Multiply a row by any nonzero constant (**scaling**).

Theorem: If the augmented matrices of two linear systems are row equivalent, then the systems have the same solution set. (i.e. The systems are equivalent!)

A key here is *structure*!

Consider the following augmented matrix. Determine if the associated system is consistent or inconsistent. If it is consistent, determine the solution set.

(a) $\left[\begin{array}{cccc} 1 & 0 & 0 & 3 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & -2 \end{array}\right]$

3 rows $\rightarrow$ 3 eqns

4 columns $\rightarrow$ 3 variables

The system is

$1x_1 + 0x_2 + 0x_3 = 3$

$0x_1 + 1x_2 + 0x_3 = 1$

$0x_1 + 0x_2 + 1x_3 = -2$

i.e. $x_1 = 3$

$x_2 = 1$

$x_3 = -2$

Obviously consistent w/ solution $\longrightarrow$

(b) $\begin{bmatrix} 1 & 0 & 2 & -5 \\ 0 & 1 & -1 & 4 \\ 0 & 0 & 0 & 3 \end{bmatrix}$

3 rows $\rightarrow$ 3 eqns

4 columns $\rightarrow$ 3 variables

It reads as

$$1x_1 + 0x_2 + 2x_3 = -5$$
$$0x_1 + 1x_2 - 1x_3 = 4$$
$$0x_1 + 0x_2 + 0x_3 = 3$$

The last equation is "$0 = 3$".

This is always false.

The system is inconsistent!

(c) $\begin{bmatrix} 1 & 0 & -2 & -4 \\ 0 & 1 & 1 & 4 \\ 0 & 0 & 0 & 0 \end{bmatrix}$

Again 3 eqn in 3 variables

$1x_1 + 0x_2 - 2x_3 = -4$

$0x_1 + 1x_2 + 1x_3 = 4$

$0x_1 + 0x_2 + 0x_3 = 0$

The last equation is "$0=0$" which is always true!

The 1st two can be stated as

$x_1 = -4 + 2x_3$

$x_2 = 4 - x_3$

x_3 - is any real number

We state this as x_3 is "free".
we call x_1 and x_2 basic variables.

This system is consistent

with infinitely many solutions,