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Section 1: Concepts and Terminology

Suppose $y = \phi(x)$ is a differentiable function. We know that $dy/dx = \phi'(x)$ is another (related) function.

For example, if $y = \cos(2x)$, then y is differentiable on $(-\infty, \infty)$. In fact,

$$\frac{dy}{dx} = -2\sin(2x).$$

Even dy/dx is differentiable with $d^2y/dx^2 = -4\cos(2x)$. Note that

$$\frac{d^2y}{dx^2} + 4y = 0.$$

The equation

$$\frac{d^2y}{dx^2} + 4y = 0.$$

is an example of a **differential equation.**

Questions: If we only started with the equation, how could we determine that $\cos(2x)$ satisfies it? Also, is $\cos(2x)$ the only possible function that y could be?

Definition

A **Differential Equation** is an equation containing the derivative(s) of one or more dependent variables, with respect to one or more indendent variables.

Independent Variable: will appear as one that derivatives are taken with respect to.

Dependent Variable: will appear as one that derivatives are taken of.

y is dependent

$$\frac{dy}{dx}$$

x is independent

$$\frac{du}{dt}$$

u is dependent

t is independent

$$\frac{dx}{dr}$$

x is dependent

r is independent

Classifications

Type: An **ordinary differential equation (ODE)** has exactly one independent variable[1]. For example

$$\frac{dy}{dx} - y^2 = 3x, \quad \text{or} \quad \frac{dy}{dt} + 2\frac{dx}{dt} = t, \quad \text{or} \quad y'' + 4y = 0$$

A **partial differential equation (PDE)** has two or more independent variables. For example

$$\frac{\partial y}{\partial t} = \frac{\partial^2 y}{\partial x^2}, \quad \text{or} \quad \frac{\partial^2 u}{\partial r^2} + \frac{1}{r}\frac{\partial u}{\partial r} + \frac{1}{r^2}\frac{\partial^2 u}{\partial \theta^2} = 0$$

∂ - partial symbol

[1] These are the subject of this course.

Classifications

Order: The order of a differential equation is the same as the highest order derivative appearing anywhere in the equation.

$$\frac{dy}{dx} - y^2 = 3x$$

1st order eqn.

$$y''' + (y')^4 = x^3$$

3rd order eqn.

$$\frac{\partial y}{\partial t} = \frac{\partial^2 y}{\partial x^2}$$

2nd order eqn.

Notations and Symbols

We'll use standard derivative notations:

$$\text{Leibniz:}\quad \frac{dy}{dx},\quad \frac{d^2y}{dx^2},\quad \ldots\quad \frac{d^ny}{dx^n},\quad \text{or}$$

$$\text{Prime \& superscripts:}\quad y',\quad y'',\quad \ldots\quad y^{(n)}.$$

Newton's **dot notation** may be used if the independent variable is time. For example if s is a position function, then

$$\text{velocity is } \frac{ds}{dt} = \dot{s},\quad \text{and acceleration is } \frac{d^2s}{dt^2} = \ddot{s}$$

Notations and Symbols

An n^{th} order ODE, with independent variable x and dependent variable y can always be expressed as an equation

$$F(x, y, y', \dots, y^{(n)}) = 0$$

where F is some real valued function of $n+2$ variables.

Normal Form: If it is possible to isolate the highest derivative term, then we can write a **normal form** of the equation

$$\frac{d^n y}{dx^n} = f(x, y, y', \dots, y^{(n-1)}).$$

Example: $y''' + (y')^4 = x^3$

Can be written as

$$\underbrace{y''' + (y')^4 - x^3}_{\text{this is } F(x, y, y', y'', y''')} = 0$$

This can be expressed in normal form

$$y''' = -(y')^4 + x^3$$

Notations and Symbols

If $n = 1$, an equation in normal form would look like

$$\frac{dy}{dx} = f(x, y).$$

If $n = 2$, an equation in normal form would look like

$$\frac{d^2y}{dx^2} = f(x, y, y').$$

Differential Form

A first order equation may appear in the form

$$M(x, y)\,dx + N(x, y)\,dy = 0$$

Either variable could be considered independent.

$$N(x,y)\,dy = -M(x,y)\,dx$$

$$\Downarrow$$

$$\frac{dy}{dx} = \frac{-M(x,y)}{N(x,y)}.$$

if $N(x,y) \neq 0$

OR

$$M(x,y)\,dx = -N(x,y)\,dy$$

$$\Downarrow$$

$$\frac{dx}{dy} = \frac{-N(x,y)}{M(x,y)}$$

if $M(x,y) \neq 0$.

Classifications

Linearity: An n^{th} order differential equation is said to be **linear** if it can be written in the form

$$a_n(x)\frac{d^n y}{dx^n} + a_{n-1}(x)\frac{d^{n-1} y}{dx^{n-1}} + \cdots + a_1(x)\frac{dy}{dx} + a_0(x)y = g(x).$$

Note that each of the coefficients $a_0, \ldots, a_n$ and the right hand side g may depend on the independent variable but not on the dependent variable or any of its derivatives.

Examples (Linear -vs- Nonlinear)

$$y'' + 4y = 0$$ linear

$a_2(x)y'' + a_1(x)y' + a_0(x)y = g(x)$

$a_2(x) = 1$ $\quad a_0(x) = 4$

$a_1(x) = 0$ $\quad g(x) = 0$

$$t^2\frac{d^2x}{dt^2} + 2t\frac{dx}{dt} - x = e^t$$

$a_2(t) = t^2$, $\quad a_0(t) = -1$

$a_1(t) = 2t$, $\quad g(t) = e^t$

$$\frac{d^3y}{dx^3} + \left(\frac{dy}{dx}\right)^4 = x^3$$ nonlinear

$$\frac{d^3y}{dx^3} + \underbrace{\left(\frac{dy}{dx}\right)^3}_{\text{nonlinear term}} \frac{dy}{dx} = x^3$$

$$\underbrace{u'' + u'}_{} = \underbrace{\cos u}_{\text{nonlinear term since } u \text{ is dependent}}$$

nonlinear term since u is dependent

Examples

Identify the independent and dependent variables. Determine the order of the equation. State whether it is linear or nonlinear.

(a) $y'' + 2ty' = \cos t + y$

Independent - t

dependent - y

2^{nd} order

$y'' + 2ty' - y = \cos t$

This is linear.

Examples

(b) $$\frac{d^3y}{dx^3} + 2y\frac{dy}{dx} = \frac{d^2y}{dx^2} + \tan(x)$$

Independent - x

dependent - y

Order 3^{rd}

$$\frac{d^3y}{dx^3} - \frac{d^2y}{dx^2} + \underbrace{2y\frac{dy}{dx}}_{\text{this is a nonlinear term}} = \tan x$$

The eqn. is nonlinear.

Examples

(c) $\ddot{\theta} + \frac{g}{\ell} \sin\theta = 0$ g and ℓ are constant

Independent time t

Dependent θ

2^{nd} order

Nonlinear due to $\sin\theta$ term.

Solution of $F(x, y, y', \ldots, y^{(n)}) = 0$ (*)

Definition: A function ϕ defined on an interval I[2] and possessing at least n continuous derivatives on I is a **solution** of (*) on I if upon substitution (i.e. setting $y = \phi(x)$) the equation reduces to an identity.

Definition: An **implicit solution** of (*) is a relation $G(x, y) = 0$ provided there exists at least one function $y = \phi$ that satisfies both the differential equation (*) and this relation.

[2] The interval is called the *domain of the solution* or the *interval of definition*.