January 12 Math 2335 sec 51 Spring 2016

Section 1.1: The Taylor Polynomial

Let's begin by considering the task of evaluating a common function

$$f(x) = e^x \quad \text{at some value} \quad a \approx 0$$

Today, we would plug the number *a* into a predefined operation on a calculator or computer. But we can ask the question

How does the machine, which performs the operations $+$, $-$, $\times$, and $\div$, evaluate such a function?

What the machine does is run algorithms to approximate answers to within an *acceptable* degree of accuracy.

Overview of Course Concepts

Over the span of the semester, we will investigate

- The use of Taylor polynomials to approximate more exotic functions;
- The errors that arise when using machines for computing, and how to minimize error;
- How to solve some equations (root finding) using various algorithms, and how to analyze the results;
- Various methods for interpolating data;
- Ways to integrate and to differentiate using numerical approximations, and
- How to solve linear systems using efficient and error reducing methods.

We begin with the use of Taylor polynomials...

Let $n \geq 1$ be an integer. A polynomial of degree n is a function of the form

$$P(x) = a_n x^n + a_{n-1} x^{n-1} + \cdots + a_1 x + a_0$$

where $a_n \neq 0$, $a_0, \ldots, a_n$ are known real numbers called coefficients.

Polynomials are very special! Evaluating a polynomial can be done using only the operations of **addition, subtraction**, and **multiplication**!

In contrast, consider other operations we take for granted such as taking roots, logarithms, exponentiating, and evaluating trigonometric functions.

Task: Approximate $e^{0.1}$ using a tangent line.

Let $f(x) = e^x$. Since 0.1 is close to zero, we consider the tangent line to the graph of f at zero.

If we call the tangent line $P_1(x)$.

P_1 has to go through the point $(0, f(0))$ and have the same slope as f @ 0.

Slope $m = f'(0)$

$f(x) = e^x \quad , \quad f'(x) = e^x$

$f(0) = e^0 = 1$ and $m = f'(0) = e^0 = 1$

point $(0,1)$ and slope $m=1$

$$P_1(x) - 1 = 1(x-0) \Rightarrow P_1(x) = x+1$$

For $x \approx 0$ $\qquad f(x) \approx P_1(x)$

$$e^{0.1} = f(0.1) \approx P_1(0.1) = 0.1 + 1 = 1.1$$

Plot of f and p_1

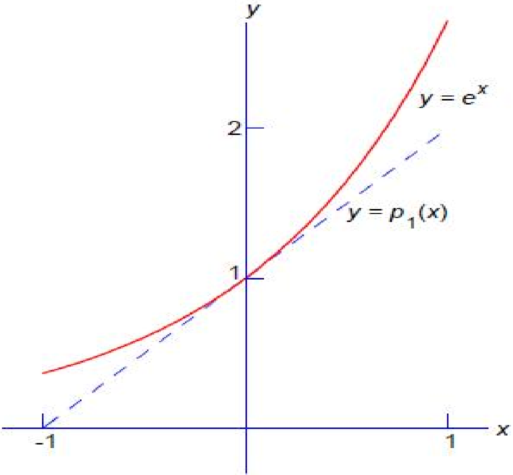


Figure: For $x \approx 0$, the two curves are very close. Note that $p_1(0) = f(0)$ and $p_1'(0) = f'(0)$.

Let's improve: Approximate $e^{0.1}$ using a quadratic.

Find a second degree polynomial $p_2(x)$ that satisfies the three conditions

$$p_2(0) = f(0), \quad p_2'(0) = f'(0), \quad \text{and} \quad p_2''(0) = f''(0).$$

A generic second degree polynomial looks like

$p_2(x) = a_2 x^2 + a_1 x + a_0$ $\qquad$ $f(x) = e^x$

$p_2'(x) = 2a_2 x + a_1$ $\qquad$ $f'(x) = e^x$

$p_2''(x) = 2a_2$ $\qquad$ $f''(x) = e^x$

$P_2(0) = a_0 \qquad f(0) = e^0 = 1 \qquad P_2(0) = f(0) \Rightarrow \quad a_0 = 1$

$P_2'(0) = a_1 \qquad f'(0) = e^0 = 1 \qquad P_2'(0) = f'(0) \Rightarrow \quad a_1 = 1$

$P_2''(0) = 2a_2 \qquad f''(0) = e^0 = 1 \qquad P_2''(0) = f''(0) \Rightarrow \quad 2a_2 = 1 \Rightarrow a_2 = \frac{1}{2}$

So $\quad P_2(x) = \frac{1}{2}x^2 + x + 1$

For $\quad x \approx 0, \quad f(x) \approx P_2(x)$

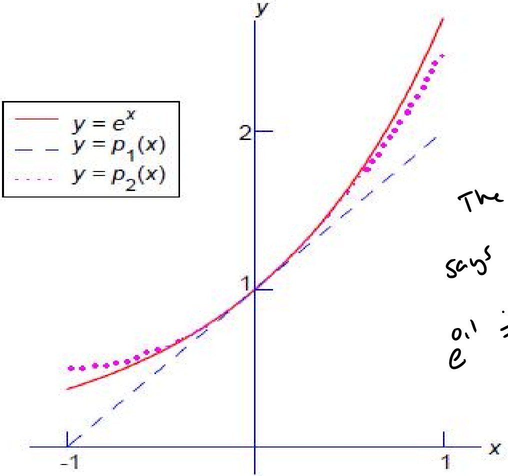
$$e^{0.1} = f(0.1) \approx P_2(0.1)$$

$$= \frac{1}{2}(0.1)^2 + 0.1 + 1$$

$$= \frac{1}{2}(0.01) + 1.1$$

$$= 0.005 + 1.1 = 1.105$$

Plot of f and p_1 and p_2



The TI 89 says

$e^{0.1} \doteq 1.105171$

Figure: Plot of f, p_1 and p_2 together.

Taylor Polynomials

Suppose that a function f has at least n continuous derivatives on an interval $\alpha < x < \beta$, and that a is some number in this interval. Determine the coefficients $c_0, c_1, \ldots, c_n$ for the polynomial

$$p_n(x) = c_0 + c_1(x-a) + c_2(x-a)^2 + \cdots + c_n(x-a)^n$$

that satisfies the $n+1$ conditions

$$\begin{array}{rcl} p_n(a) & = & f(a) \\ p_n'(a) & = & f'(a) \\ p_n''(a) & = & f''(a) \\ & \vdots & \\ p_n^{(n)}(a) & = & f^{(n)}(a). \end{array}$$

$$P_n(a) = c_0 + c_1(a-a) + c_2(a-a)^2 + \dots + c_n(a-a)^n = c_0 = f(a)$$

So $\boxed{c_0 = f(a)}$

$$P_n'(x) = c_1 + 2c_2(x-a) + 3c_3(x-a)^2 + \dots + nc_n(x-a)^{n-1}$$

$$P_n'(a) = c_1 + 0 \cdots \quad \Rightarrow \quad P_n'(a) = c_1 = f'(a)$$

$$\boxed{c_1 = f'(a)}$$

$$P_n''(x) = 2c_2 + 3\cdot 2c_3(x-a) + 4\cdot 3a_4(x-a)^2 + \dots + n(n-1)c_n(x-a)^{n-2}$$

$$P_n''(a) = 2c_2 = f''(a) \quad \Rightarrow \quad \boxed{c_2 = \frac{f''(a)}{2} = \frac{f''(a)}{2\cdot 1}}$$

$$P_n'''(x) = 3\cdot 2\, c_3 + 4\cdot 3\cdot 2\, c_4 (x-a) + 5\cdot 4\cdot 3\, c_5 (x-a)^2 + \cdots + n(n-1)(n-2)\, c_n (x-a)^{n-3}$$

$$P_n'''(a) = 3\cdot 2\, c_3 = f'''(a) \Rightarrow \boxed{c_3 = \frac{f'''(a)}{3\cdot 2} = \frac{f'''(a)}{3\cdot 2\cdot 1} = \frac{f'''(a)}{3!}}$$

We deduce

$$\boxed{c_k = \frac{f^{(k)}(a)}{k!}}$$

Definition: Taylor Polynomial

Suppose f has at least n continuous derivatives on the interval (α, β) and that a is a point in this interval. The Taylor polynomial of degree n centered at a for the function f is

$$p_n(x) = \frac{f(a)}{0!} + \frac{f'(a)}{1!}(x-a) + \frac{f''(a)}{2!}(x-a)^2 + \cdots + \frac{f^{(n)}(a)}{n!}(x-a)^n$$

Notation: $j!$ is read "j factorial", where $0! = 1$ and $j! = 1 \cdot 2 \cdot 3 \cdots j$. We'll be careful to denote derivatives with parentheses $f^{(n)}$ indicates an n^{th} derivative as opposed to f^n which is read as a power.

Example

Find the Taylor polynomial of degree 3 for $f(x) = \ln x$ centered at $a = 2$.

For $a = 2$

$$P_3(x) = \frac{f(2)}{0!} + \frac{f'(2)}{1!}(x-2) + \frac{f''(2)}{2!}(x-2)^2 + \frac{f'''(2)}{3!}(x-2)^3$$

$f(x) = \ln x$ $\quad$ $f(2) = \ln 2$ $\quad$ $0! = 1$

$f'(x) = \frac{1}{x}$ $\quad$ $f'(2) = \frac{1}{2}$ $\quad$ $1! = 1$

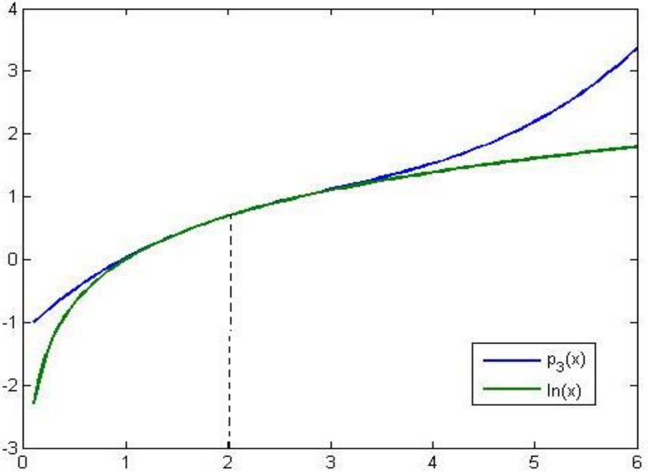
$f''(x) = \frac{-1}{x^2}$ $\quad$ $f''(2) = \frac{-1}{2^2} = \frac{-1}{4}$ $\quad$ $2! = 2$

$f'''(x) = \frac{2}{x^3}$ $\quad$ $f'''(2) = \frac{2}{2^3} = \frac{1}{2^2} = \frac{1}{4}$ $\quad$ $3! = 6$

So

$$P_3(x) = \ln 2 + \frac{1}{2}(x-2) - \frac{1}{8}(x-2)^2 + \frac{1}{24}(x-2)^3$$

Plot of $\ln x$ and p_3 centered at $a = 2$



Figure