January 13 MATH 1112 sec. 54 Spring 2020

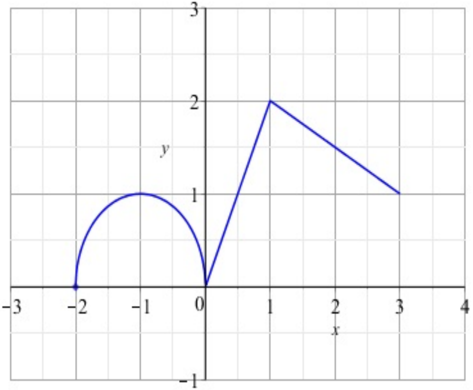
Basic Graph Transformations

From a small library of known function plots, we can graph a variety of functions if they can be determined as simple tranformations. We'll consider the following transformations:

- **Translations** shifting a graph up or down (vertical) or to the left or right (horizontal)
- **Reflections** taking the *mirror* image in the x or y axis
- **Scaling** stretching or shriking a graph in either of the vertical or horizontal orientations

Let's Explore Some Translations

$y = f(x)$



x	$f(x)$
-2	0
-1	1
0	0
1	2
2	$\frac{3}{2}$
3	1

Figure: The plot of $y = f(x)$ is shown along with a table of some select points. We will consider the effect on the graph of various translations on this function.

Vertical Translation: $y = f(x) + b$ or $y = f(x) - b$

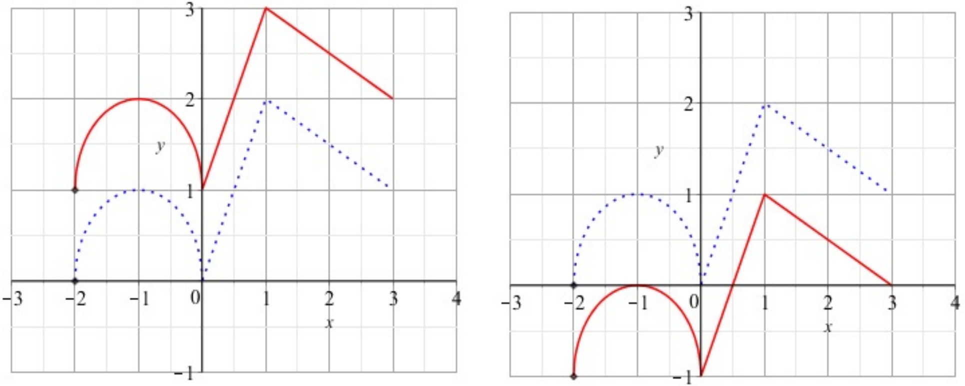


Figure: Left: $y = f(x)$ (blue dots), compared to $y = f(x) + 1$ (red)
Right: $y = f(x)$ (blue dots), compared to $y = f(x) - 1$ (red)

Horizontal Translation: $y = f(x - d)$ or $y = f(x + d)$

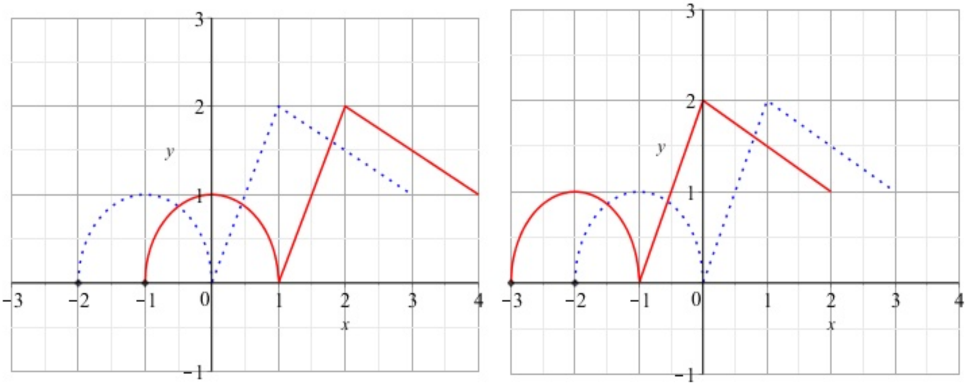


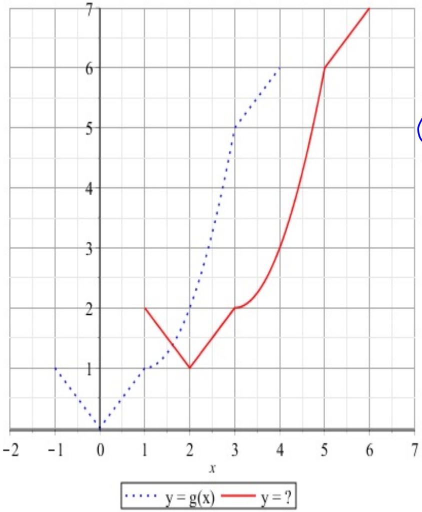
Figure: Left: $y = f(x)$ (blue dots), compared to $y = f(x - 1)$ (red)
Right: $y = f(x)$ (blue dots), compared to $y = f(x + 1)$ (red)

Vertical and Horizontal Translations

For $b > 0$ and $d > 0$

- the graph of $y = f(x) + b$ is the graph of $y = f(x)$ shifted up b units,
- the graph of $y = f(x) - b$ is the graph of $y = f(x)$ shifted down b units,
- the graph of $y = f(x - d)$ is the graph of $y = f(x)$ shifted right d units,
- the graph of $y = f(x + d)$ is the graph of $y = f(x)$ shifted left d units,

Question



The blue dotted curve is y = g(x). The red solid curve is the graph of y =

(a) $g(x-2)+1$

(b) $g(x+2)+1$

(c) $g(x-2)-1$

(d) $g(x+2)-1$

(e) can't be determined without more information

Reflections: $y = f(-x)$ or $y = -f(x)$

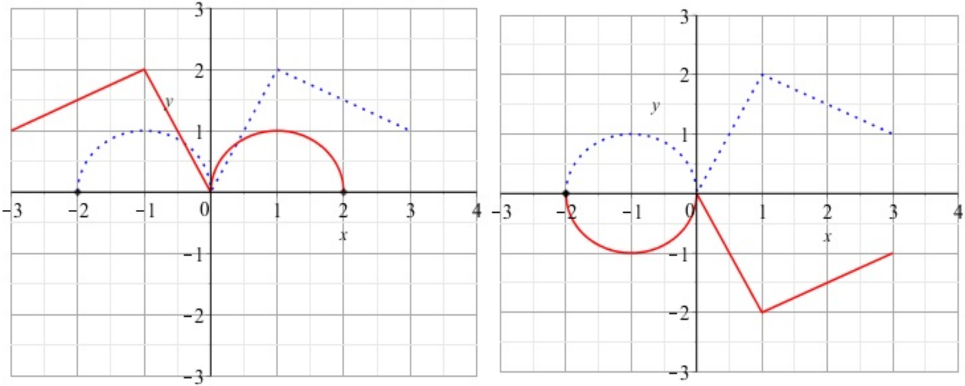


Figure: Left: $y = f(x)$ (blue dots), compared to $y = f(-x)$ (red)
Right: $y = f(x)$ (blue dots), compared to $y = -f(x)$ (red)

Reflection in the coordinate axes

The graph of $y = f(-x)$ is the reflection of the graph of $y = f(x)$ across the y-axis.

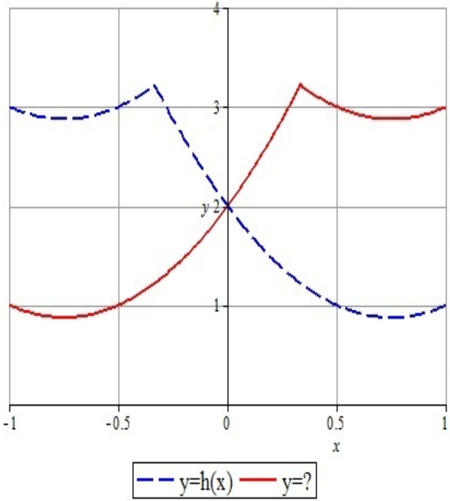
The graph of $y = -f(x)$ is the reflection of the graph of $y = f(x)$ across the x-axis.

Note that if (a, b) is a point on the graph of $y = f(x)$, then

(1) the point $(-a, b)$ is on the graph of $y = f(-x)$, and

(2) the point $(a, -b)$ is on the graph of $y = -f(x)$.

Question



The blue dashed curve is $y = h(x)$. The solid red curve is a plot of

(a) $y = -h(x)$

(b) $y = h(x - 2)$

(c) $y = h(-x)$

(d) Can't be determined without more information.

Stretching and Shrinking

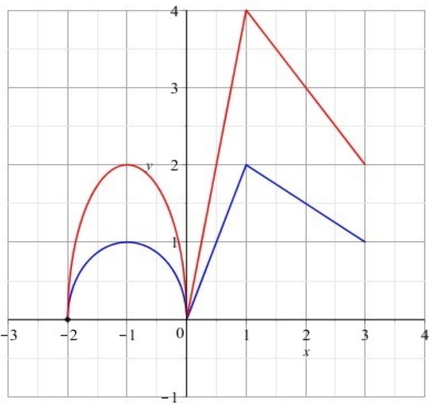
Since we already know that introducing a minus sign as in $f(-x)$ and $-f(x)$ results in a reflection, let's consider a positive number a and investigate the relationship between the graph of $y = f(x)$ and each of

$$y = af(x), \quad \text{and} \quad y = f(ax).$$

The first outcome depends on whether $a > 1$ or $0 < a < 1$.

Why aren't we bothering with the case $a = 1$?

Vertical Stretch or Shrink: $y = af(x)$



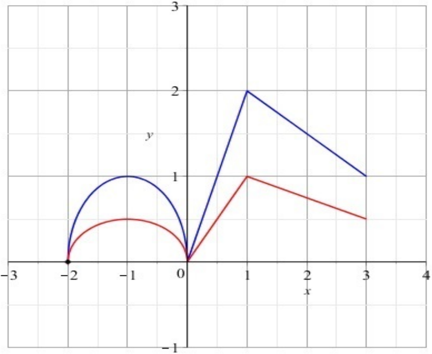
x	$f(x)$
-2	0
-1	1
0	0
1	2
2	$\frac{3}{2}$
3	1

x	$2f(x)$
-2	0
-1	2
0	0
1	4
2	3
3	2

doubled y's

Figure: $y = f(x)$ is in blue, and $y = 2f(x)$ is in red. Since $a = 2 > 1$, the graph is stretched vertically.

Vertical Stretch or Shrink: $y = af(x)$



x	$f(x)$
-2	0
-1	1
0	0
1	2
2	$\frac{3}{2}$
3	1

x	$\frac{1}{2}f(x)$
-2	0
-1	1/2
0	0
1	1
2	3/4
3	1/2

halved y's

Figure: $y = f(x)$ is in blue, and $y = \frac{1}{2}f(x)$ is in red. Since $a = \frac{1}{2} < 1$, the graph is shrinked vertically.

Vertical Stretch or Shrink: $y = af(x)$

The graph of $y = af(x)$ is obtained from the graph of $y = f(x)$. If $a > 0$, then

$y = af(x)$ is stretched vertically if $a > 1$, and
$y = af(x)$ is shrunk (a.k.a. compressed) vertically if $0 < a < 1$.

If $a < 0$, then the stretch ($|a| > 1$) or shrink ($0 < |a| < 1$) is combined with a reflection in the x-axis.

Horizontal Stretch or Shrink: $y = f(ax)$

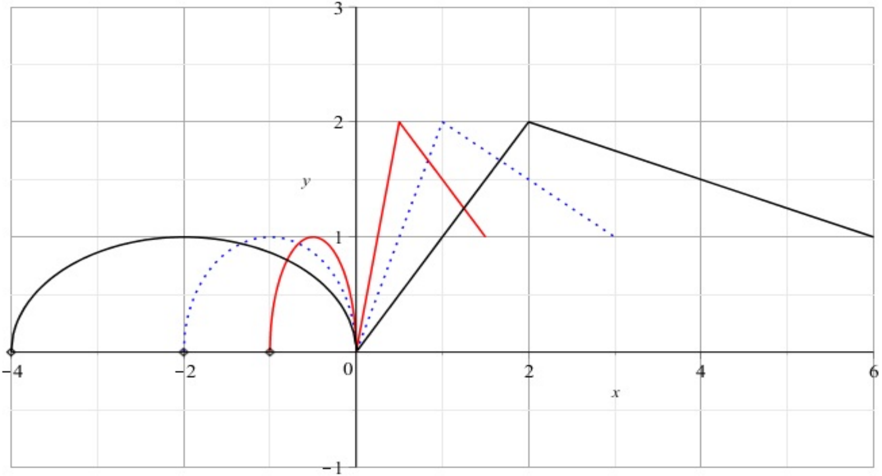


Figure: $y = f(x)$ is in blue dots. The compressed red curve is $y = f(2x)$, and the stretched black curve is $y = f\left(\frac{1}{2}x\right)$.

Horizontal Stretch or Shrink: $y = f(ax)$

The examples given generalize except that we did not consider an example with $a < 0$. This combines the stretch/shrink with a reflection. We have the following result:

The graph of $y = f(ax)$ is obtained from the graph of $y = f(x)$. If $a > 0$, then

$y = f(ax)$ is shrunk (a.k.a. compressed) horizontally if $a > 1$, and
$y = f(ax)$ is stretched horizontally if $0 < a < 1$.

If $a < 0$, then the shrink ($|a| > 1$) or stretch ($0 < |a| < 1$) is combined with a reflection in the y-axis.

Symmetry

For a function or a relation given in terms of algebraic expressions, we can check for symmetry:

- ***y*-axis (Even)** if replacing (x, y) with $(-x, y)$ results in the same formula (e.g. $f(-x) = f(x)$)
- **Origin (Odd):** if replacing (x, y) with $(-x, -y)$ results in the same foruma (e.g. $f(-x) = -f(x)$)
- ***x*-axis:** if replacing (x, y) with $(x, -y)$ results in the same formula.

Even and Odd Functions

Definition: If a function f is called an **even function** if

$$f(-x) = f(x)$$

for each x in its domain. We can say that such a function has **even symmetry**.

the graph of f is its own reflection in the y-axis!

Definition: If a function f is called an **odd function** if

$$f(-x) = -f(x)$$

for each x in its domain. We can say that such a function has **odd symmetry**.

the reflection of f in the y-axis is its reflection in the x-axis!

Even Symmetry

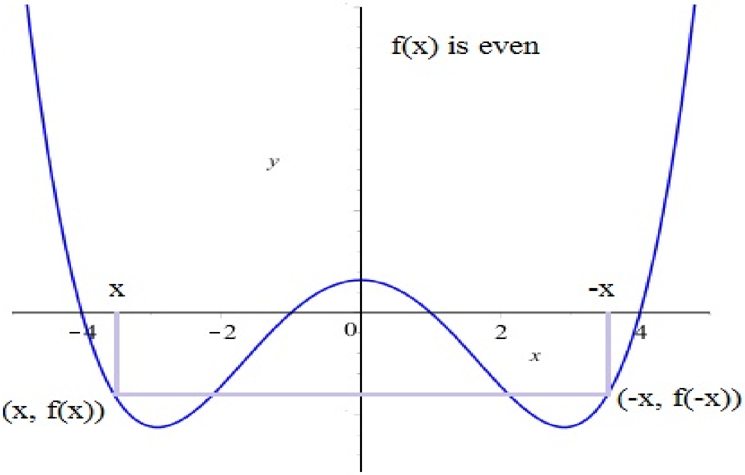


Figure: Graph is its reflection in the *y*-axis

Odd Symmetry

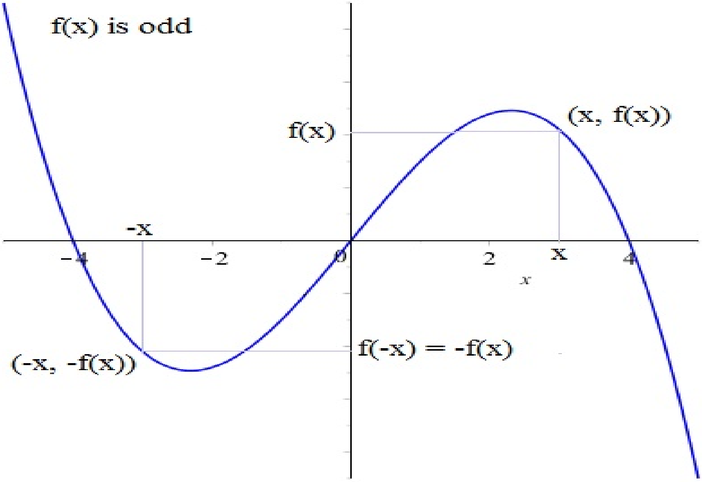


Figure: The point $(x, f(x))$ has reflection $(-x, -f(x))$ on the graph.

x-axis Symmetry

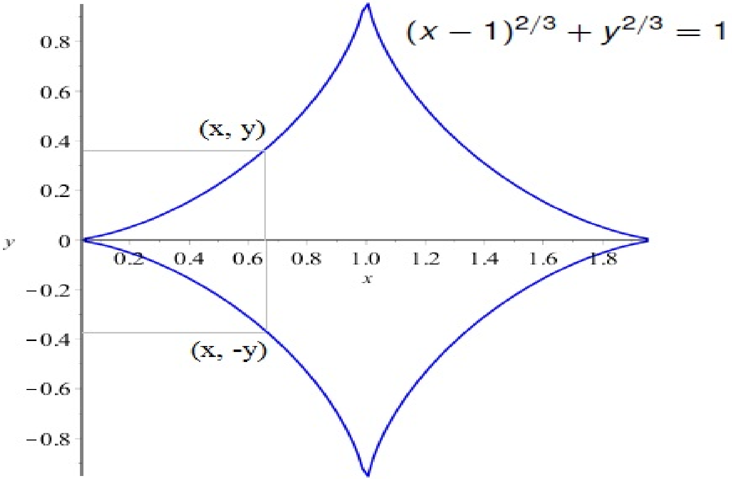


Figure: Graph is its reflection in the x-axis.