January 13 Math 3260 sec. 55 Spring 2020

Section 1.2: Row Reduction and Echelon Forms

We'll use the following notation for the three types of elementary row operations. We let R_i represent the ith row in a matrix.

▶ Swap rows i and j:

$$R_i \leftrightarrow R_j$$

▶ Scale row i by k:

$$kR_i \rightarrow R_i$$

▶ Replace row j with the sum of itself and k times row i:

$$kR_i + R_j \rightarrow R_j$$

Theorem: If the augmented matrices of two linear systems are row equivalent, then the systems are equivalent.

Echelon Forms

Definition: (ref) A matrix is in **echelon form** (a.k.a. **row echelon form**) if the following properties hold

   i Any row of all zeros are at the bottom.
   ii The first nonzero number (called the *leading entry*) in a row is to the right of the first nonzero number in all rows above it.
   iii All entries below a leading entry are zeros.

Definition: (rref) A matrix is in **reduced echelon form** (a.k.a. **reduced row echelon form**) if it is in echelon form and the following additional properties hold

   iv The leading entry of each row is 1 (called a *leading* 1), and
   v each leading 1 is the only nonzero entry in its column.

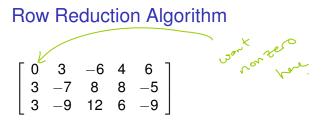# Using Row Operations to Obtain an ref or rref

We'll use a fairly methodical approach to obtaining an ref or an rref from a given matrix. The basic process is as follows:

1. **Matrix → ref :** Work towards an ref starting on the top left, and working down and to the right. Get leading entries and zero below them.

2. **ref → rref:** Work from bottom right-most leading entry, obtain zeros above each leading entry moving upward and to the left.

Row Reduction Algorithm

$$\begin{bmatrix} 0 & 3 & -6 & 4 & 6 \\ 3 & -7 & 8 & 8 & -5 \\ 3 & -9 & 12 & 6 & -9 \end{bmatrix}$$
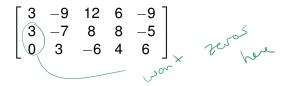
won't nonzero here.

Step 1: If the first column is not all zero[1], get a leading entry in the top left position. Row swap if necessary. We'll do $R_1 \leftrightarrow R_3$

$$R_1 \leftrightarrow R_3 \qquad \begin{bmatrix} 3 & -9 & 12 & 6 & -9 \\ 3 & -7 & 8 & 8 & -5 \\ 0 & 3 & -6 & 4 & 6 \end{bmatrix}$$

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[1] If the first column is all zero, move to the second column.

Row Reduction Algorithm

$$\begin{bmatrix} 3 & -9 & 12 & 6 & -9 \\ 3 & -7 & 8 & 8 & -5 \\ 0 & 3 & -6 & 4 & 6 \end{bmatrix}$$

want zeros here

Step 2: Use row operations to get zeros in all entries below the first leading entry. (Use replacements to get zeros below the leading entry.)

$-R_1 + R_2 \rightarrow R_2$

$$\begin{bmatrix} 3 & -9 & 12 & 6 & -9 \\ 0 & 2 & -4 & 2 & 4 \\ 0 & 3 & -6 & 4 & 6 \end{bmatrix}$$

Row Reduction Algorithm

$$\begin{bmatrix} 3 & -9 & 12 & 6 & -9 \\ 0 & 2 & -4 & 2 & 4 \\ 0 & 3 & -6 & 4 & 6 \end{bmatrix}$$

Step 3: Ignore the row and column with the leading entry, all rows above it, and all columns to its left, and repeat steps.

Let's scale the 2nd row and get a zero below the 2nd leading entry

$$\frac{1}{2} R_2 \to R_2$$

$$\begin{bmatrix} 3 & -9 & 12 & 6 & -9 \\ 0 & 1 & -2 & 1 & 2 \\ 0 & 3 & -6 & 4 & 6 \end{bmatrix}$$

Row Reduction Algorithm

$$-3R_2 + R_3 \rightarrow R_3$$

$$\begin{bmatrix} 3 & -9 & 12 & 6 & -9 \\ 0 & 1 & -2 & 1 & 2 \\ 0 & 0 & 0 & 1 & 0 \end{bmatrix}$$

This is an ref

Row Reduction Algorithm: From ref to rref.

$$\begin{bmatrix} 3 & -9 & 12 & 6 & -9 \\ 0 & 1 & -2 & 1 & 2 \\ 0 & 0 & 0 & 1 & 0 \end{bmatrix}$$

need zeros here

Step 4: Starting from the right most leading entry, use row operations to obtain zeros above the leading entries. Work from right to left, bottom up, and scale each leading entry to make it a 1 (a **leading one**).

$-R_3 + R_2 \rightarrow R_2$ and $-6R_3 + R_1 \rightarrow R_1$

$$\begin{bmatrix} 3 & -9 & 12 & 0 & -9 \\ 0 & 1 & -2 & 0 & 2 \\ 0 & 0 & 0 & 1 & 0 \end{bmatrix}$$

now, we need a zero here

Row Reduction Algorithm

$$9R_2 + R_1 \to R_1$$

I need 1 here

$$\begin{bmatrix} 3 & 0 & -6 & 0 & 9 \\ 0 & 1 & -2 & 0 & 2 \\ 0 & 0 & 0 & 1 & 0 \end{bmatrix}$$

$$\tfrac{1}{3} R_1 \to R_1$$

$$\begin{bmatrix} 1 & 0 & -2 & 0 & 3 \\ 0 & 1 & -2 & 0 & 2 \\ 0 & 0 & 0 & 1 & 0 \end{bmatrix}$$