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Section 2: Initial Value Problems

An initial value problem consists of an ODE with a certain type of additional conditions.

Solve the equation [1]

$$\frac{d^n y}{dx^n} = f(x, y, y', \dots, y^{(n-1)}) \tag{1}$$

subject to the *initial conditions*

$$y(x_0) = y_0, \quad y'(x_0) = y_1, \quad \dots, y^{(n-1)}(x_0) = y_{n-1}. \tag{2}$$

The problem (1)–(2) is called an *initial value problem* (IVP).

[1] on some interval I containing x_0.

Example

Part 1: Verify that $x = c_1 \cos(2t) + c_2 \sin(2t)$ is a 2-parameter family of solutions of the ODE

$$x'' + 4x = 0.$$

$x = c_1 \cos(2t) + c_2 \sin(2t)$

$x' = -2c_1 \sin(2t) + 2c_2 \cos(2t)$

$x'' = -4c_1 \cos(2t) - 4c_2 \sin(2t)$

Substitute into the ODE

$x'' + 4x = \left(-4c_1 \cos(2t) - 4c_2 \sin(2t)\right) + 4\left(c_1 \cos(2t) + c_2 \sin(2t)\right) =$

$$= -4c_1\cos(2t) - 4c_2\sin(2t) + 4c_1\cos(2t) + 4c_2\sin(2t)$$

$$= \cos(2t)\left(-4c_1 + 4c_1\right) + \sin(2t)\left(-4c_2 + 4c_2\right)$$

$$= 0 + 0 = 0$$

So $x = c_1\cos(2t) + c_2\sin(2t)$ is a solution for any pair c_1, c_2.

Example

Part 2: Find a solution of the IVP

$$x'' + 4x = 0, \quad x\left(\frac{\pi}{2}\right) = -1, \quad x'\left(\frac{\pi}{2}\right) = 4$$

We know that $x = C_1 \cos(2t) + C_2 \sin(2t)$. We impose the conditions $x\left(\frac{\pi}{2}\right) = -1$ and $x'\left(\frac{\pi}{2}\right) = 4$.

$$x' = -2C_1 \sin(2t) + 2C_2 \cos(2t).$$

$$x\left(\frac{\pi}{2}\right) = C_1 \cos\left(2\cdot\frac{\pi}{2}\right) + C_2 \sin\left(2\cdot\frac{\pi}{2}\right) = -1$$

$$C_1(-1) + C_2(0) = -1 \implies C_1 = 1$$

$$x'\left(\frac{\pi}{2}\right) = -2c_1 \sin\left(2\cdot\frac{\pi}{2}\right) + 2c_2 \cos\left(2\cdot\frac{\pi}{2}\right) = 4$$

$$-2c_1(0) + 2c_2(-1) = 4 \implies c_2 = -2$$

$$\boxed{\text{The solution to the IVP is} \quad x = \cos(2t) - 2\sin(2t)}$$

Existence and Uniqueness

Two important questions we can always pose (and sometimes answer) are

(1) Does an IVP have a solution? (existence) and

(2) If it does, is there just one? (uniqueness)

Hopefully it's obvious that we can't solve $\underbrace{\left(\frac{dy}{dx}\right)^2 + 1}_{\geq 1} = \underbrace{-y^2}_{\leq 0}.$

Uniqueness

Consider the IVP

$$\frac{dy}{dx} = x\sqrt{y} \quad y(0) = 0$$

Verify that $y = \frac{x^4}{16}$ is a solution of the IVP. And find a second solution of the IVP by clever guessing.

Solving an IVP means satisfying a differential equation AND satisfying an initial condition.

Show that $y = \frac{x^4}{16}$ satisfies the initial condition:

$y(0) = \frac{0^4}{16} = \frac{0}{16} = 0$, when $x=0, y=0$. So this satisfies the initial condition.

$$\frac{dy}{dx} = x\sqrt{y} \quad y(0) = 0$$

Show that $y = \frac{x^4}{16}$ **solves the differential equation:**

$y = \frac{x^4}{16} \Rightarrow \frac{dy}{dx} = \frac{4x^3}{16} = \frac{x^3}{4}$ and $\sqrt{y} = \sqrt{\frac{x^4}{16}} = \frac{|x^2|}{4} = \frac{x^2}{4}$

$$\frac{dy}{dx} = \frac{x^3}{4} = x\sqrt{y} = x\left(\frac{x^2}{4}\right) = \frac{x^3}{4}$$

that is, $\frac{dy}{dx}$ does equal $x\sqrt{y}$.

$y = \frac{x^4}{16}$ solves the IVP.

$$\frac{dy}{dx} = x\sqrt{y} \quad y(0) = 0$$

Find another solution to the IVP. (Hint: Think about really simple functions like a constant function.)

Another solution is $y(x) = 0$.

If $y=0$, then $\frac{dy}{dx} = 0$ and $\sqrt{y} = \sqrt{0} = 0$

Section 3: Separation of Variables

The simplest type of equation we could encounter would be of the form

$$\frac{dy}{dx} = g(x).$$

For example, solve the ODE

$$\frac{dy}{dx} = 4e^{2x} + 1. \quad \Longrightarrow \quad y = \int (4e^{2x} + 1)\, dx$$

$$= 2e^{2x} + x + C$$

This is a one parameter family of solutions.

Separable Equations

Definition: The first order equation $y' = f(x, y)$ is said to be **separable** if the right side has the form

$$f(x, y) = g(x)h(y).$$

That is, a separable equation is one that has the form

$$\frac{dy}{dx} = g(x)h(y).$$

Determine which (if any) of the following are separable.

(a) $\dfrac{dy}{dx} = x^3 y$

This is separable

$g(x) = x^3 \quad , \quad h(y) = y$

(b) $\dfrac{dy}{dx} = 2x + y$

This is not separable.

Note $\quad 2x + y = x\left(2 + \dfrac{y}{x}\right)$